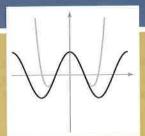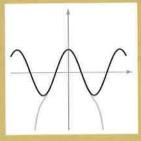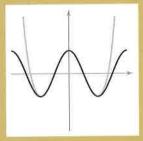# 9 Infinite Series

A *ballista* was used as a portable rock-throwing machine. The propulsion mechanism was similar in appearance to a crossbow. Skilled artillerymen aimed and fired the ballista entirely by eye. What type of projectile path do you think these artillerymen preferred–a high, arching trajectory or a low, relatively level trajectory? Why?

Maclaurin *polynomials* approximate a given function in an interval around $x = 0$. As you add terms to the Maclaurin polynomial, it becomes a better approximation of the given function near $x = 0$. In Section 9.10, you will see that a Maclaurin *series* is equivalent to the given function (under suitable conditions).

## Sequences

- List the terms of a sequence.
- Determine whether a sequence converges or diverges.
- Write a formula for the $n$th term of a sequence.
- Use properties of monotonic sequences and bounded sequences.

### EXPLORATION

***Finding Patterns*** Describe a pattern for each of the following sequences. Then use your description to write a formula for the $n$th term of each sequence. As $n$ increases, do the terms appear to be approaching a limit? Explain your reasoning.

**a.** $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

**b.** $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

**c.** $10, \frac{10}{3}, \frac{10}{6}, \frac{10}{10}, \frac{10}{15}, \ldots$

**d.** $\frac{1}{4}, \frac{4}{9}, \frac{9}{16}, \frac{16}{25}, \frac{25}{36}, \ldots$

**e.** $\frac{3}{7}, \frac{5}{10}, \frac{7}{13}, \frac{9}{16}, \frac{11}{19}, \ldots$

**NOTE** Occasionally, it is convenient to begin a sequence with $a_0$, so that the terms of the sequence become

$a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$

**STUDY TIP** Some sequences are defined recursively. To define a sequence recursively, you need to be given one or more of the first few terms. All other terms of the sequence are then defined using previous terms, as shown in Example 1(d).

## Sequences

In mathematics, the word "sequence" is used in much the same way as in ordinary English. To say that a collection of objects or events is *in sequence* usually means that the collection is ordered so that it has an identified first member, second member, third member, and so on.

Mathematically, a **sequence** is defined as a function whose domain is the set of positive integers. Although a sequence is a function, it is common to represent sequences by subscript notation rather than by the standard function notation. For instance, in the sequence

$$
\begin{array}{ccccccc}
1, & 2, & 3, & 4, & \ldots, & n, & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
a_1, & a_2, & a_3, & a_4, & \ldots, & a_n, & \ldots
\end{array}
$$

Sequence

1 is mapped onto $a_1$, 2 is mapped onto $a_2$, and so on. The numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ are the **terms** of the sequence. The number $a_n$ is the **$n$th term** of the sequence, and the entire sequence is denoted by $\{a_n\}$.

### EXAMPLE I Listing the Terms of a Sequence

**a.** The terms of the sequence $\{a_n\} = \{3 + (-1)^n\}$ are

$$3 + (-1)^1, \ 3 + (-1)^2, \ 3 + (-1)^3, \ 3 + (-1)^4, \ldots$$
$$2, \qquad 4, \qquad 2, \qquad 4, \qquad \ldots$$

**b.** The terms of the sequence $\{b_n\} = \left\{ \dfrac{n}{1 - 2n} \right\}$ are

$$\frac{1}{1 - 2 \cdot 1}, \ \frac{2}{1 - 2 \cdot 2}, \ \frac{3}{1 - 2 \cdot 3}, \ \frac{4}{1 - 2 \cdot 4}, \ \cdots$$
$$-1, \qquad -\frac{2}{3}, \qquad -\frac{3}{5}, \qquad -\frac{4}{7}, \qquad \cdots$$

**c.** The terms of the sequence $\{c_n\} = \left\{ \dfrac{n^2}{2^n - 1} \right\}$ are

$$\frac{1^2}{2^1 - 1}, \ \frac{2^2}{2^2 - 1}, \ \frac{3^2}{2^3 - 1}, \ \frac{4^2}{2^4 - 1}, \ \cdots$$
$$\frac{1}{1}, \qquad \frac{4}{3}, \qquad \frac{9}{7}, \qquad \frac{16}{15}, \qquad \cdots$$

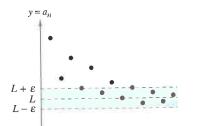**d.** The terms of the **recursively defined** sequence $\{d_n\}$, where $d_1 = 25$ and $d_{n+1} = d_n - 5$ are

$$25, \ 25 - 5 = 20, \ 20 - 5 = 15, \ 15 - 5 = 10, \ldots$$

## Limit of a Sequence

The primary focus of this chapter concerns sequences whose terms approach limiting values. Such sequences are said to **converge.** For instance, the sequence $\{1/2^n\}$

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

converges to 0, as indicated in the following definition.



$y = a_n$

$L + \varepsilon$
$L$
$L - \varepsilon$

1 2 3 4 5 6  $\cdots$  $M$

For $n > M$, the terms of the sequence all lie within $\varepsilon$ units of $L$.
**Figure 9.1**

---

**Definition of the Limit of a Sequence**

Let $L$ be a real number. The **limit** of a sequence $\{a_n\}$ is $L$, written as

$$\lim_{n \to \infty} a_n = L$$

if for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. If the limit $L$ of a sequence exists, then the sequence **converges** to $L$. If the limit of a sequence does not exist, then the sequence **diverges.**

---

Graphically, this definition says that eventually (for $n > M$ and $\varepsilon > 0$) the terms of a sequence that converges to $L$ will lie within the band between the lines $y = L + \varepsilon$ and $y = L - \varepsilon$, as shown in Figure 9.1.

If a sequence $\{a_n\}$ agrees with a function $f$ at every positive integer, and if $f(x)$ approaches a limit $L$ as $x \to \infty$, the sequence must converge to the same limit $L$.

---

**THEOREM 9.1    Limit of a Sequence**

Let $L$ be a real number. Let $f$ be a function of a real variable such that

$$\lim_{x \to \infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer $n$, then

$$\lim_{n \to \infty} a_n = L.$$

---

### EXAMPLE 2    Finding the Limit of a Sequence

**NOTE**  There are different ways in which a sequence can fail to have a limit. One way is that the terms of the sequence increase without bound or decrease without bound. These cases are written symbolically as follows.

Terms increase without bound:

$$\lim_{n \to \infty} a_n = \infty$$

Terms decrease without bound:

$$\lim_{n \to \infty} a_n = -\infty$$

Find the limit of the sequence whose $n$th term is

$$a_n = \left(1 + \frac{1}{n}\right)^n.$$

**Solution**   In Theorem 5.15, you learned that

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e.$$

So, you can apply Theorem 9.1 to conclude that

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

$$= e.$$

The following properties of limits of sequences parallel those given for limits of functions of a real variable in Section 1.3.

---

**THEOREM 9.2   Properties of Limits of Sequences**

Let $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = K$.

**1.** $\lim\limits_{n\to\infty} (a_n \pm b_n) = L \pm K$

**2.** $\lim\limits_{n\to\infty} ca_n = cL$, $c$ is any real number

**3.** $\lim\limits_{n\to\infty} (a_n b_n) = LK$

**4.** $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{L}{K}$, $b_n \neq 0$ and $K \neq 0$

---

**EXAMPLE 3   Determining Convergence or Divergence**

**a.** Because the sequence $\{a_n\} = \{3 + (-1)^n\}$ has terms

$$2, 4, 2, 4, \ldots \qquad \text{\small\color{red}See Example 1(a), page 594.}$$

that alternate between 2 and 4, the limit

$$\lim_{n\to\infty} a_n$$

does not exist. So, the sequence diverges.

**b.** For $\{b_n\} = \left\{\dfrac{n}{1 - 2n}\right\}$, divide the numerator and denominator by $n$ to obtain

$$\lim_{n\to\infty} \frac{n}{1 - 2n} = \lim_{n\to\infty} \frac{1}{(1/n) - 2} = -\frac{1}{2} \qquad \text{\small\color{red}See Example 1(b), page 594.}$$

which implies that the sequence converges to $-\frac{1}{2}$.

---

**EXAMPLE 4   Using L'Hôpital's Rule to Determine Convergence**

Show that the sequence whose $n$th term is $a_n = \dfrac{n^2}{2^n - 1}$ converges.

**Solution**   Consider the function of a real variable

$$f(x) = \frac{x^2}{2^x - 1}.$$

Applying L'Hôpital's Rule twice produces

$$\lim_{x\to\infty} \frac{x^2}{2^x - 1} = \lim_{x\to\infty} \frac{2x}{(\ln 2)2^x} = \lim_{x\to\infty} \frac{2}{(\ln 2)^2 2^x} = 0.$$

Because $f(n) = a_n$ for every positive integer, you can apply Theorem 9.1 to conclude that

$$\lim_{n\to\infty} \frac{n^2}{2^n - 1} = 0. \qquad \text{\small\color{red}See Example 1(c), page 594.}$$

So, the sequence converges to 0.

**TECHNOLOGY**   Use a graphing utility to graph the function in Example 4. Notice that as $x$ approaches infinity, the value of the function gets closer and closer to 0. If you have access to a graphing utility that can generate terms of a sequence, try using it to calculate the first 20 terms of the sequence in Example 4. Then view the terms to observe numerically that the sequence converges to 0.

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*indicates that in the* HM mathSpace® *CD-ROM and the online* Eduspace® *system for this text, you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

The symbol $n!$ (read "$n$ factorial") is used to simplify some of the formulas developed in this chapter. Let $n$ be a positive integer; then $n$ **factorial** is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n - 1) \cdot n.$$

As a special case, **zero factorial** is defined as $0! = 1$. From this definition, you can see that $1! = 1$, $2! = 1 \cdot 2 = 2$, $3! = 1 \cdot 2 \cdot 3 = 6$, and so on. Factorials follow the same conventions for order of operations as exponents. That is, just as $2x^3$ and $(2x)^3$ imply different orders of operations, $2n!$ and $(2n)!$ imply the following orders.

$$2n! = 2(n!) = 2(1 \cdot 2 \cdot 3 \cdot 4 \cdots n)$$

and

$$(2n)! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n \cdot (n + 1) \cdots 2n$$

Another useful limit theorem that can be rewritten for sequences is the Squeeze Theorem from Section 1.3.
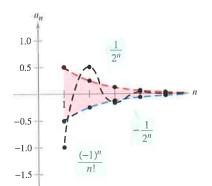
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**THEOREM 9.3    Squeeze Theorem for Sequences**

If

$$\lim_{n \to \infty} a_n = L = \lim_{n \to \infty} b_n$$

and there exists an integer $N$ such that $a_n \le c_n \le b_n$ for all $n > N$, then

$$\lim_{n \to \infty} c_n = L.$$

---



For $n \ge 4$, $(-1)^n / n!$ is squeezed between $-1/2^n$ and $1/2^n$.
**Figure 9.2**

**EXAMPLE 5    Using the Squeeze Theorem**

Show that the sequence $\{c_n\} = \left\{ (-1)^n \dfrac{1}{n!} \right\}$ converges, and find its limit.

**Solution**    To apply the Squeeze Theorem, you must find two convergent sequences that can be related to the given sequence. Two possibilities are $a_n = -1/2^n$ and $b_n = 1/2^n$, both of which converge to 0. By comparing the term $n!$ with $2^n$, you can see that

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots n = 24 \cdot \underbrace{5 \cdot 6 \cdots n}_{n - 4 \text{ factors}} \qquad {\color{red}(n \ge 4)}$$

and

$$2^n = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdots 2 = 16 \cdot \underbrace{2 \cdot 2 \cdots 2}_{n - 4 \text{ factors}}. \qquad {\color{red}(n \ge 4)}$$

This implies that for $n \ge 4$, $2^n < n!$, and you have

$$\frac{-1}{2^n} \le (-1)^n \frac{1}{n!} \le \frac{1}{2^n}, \quad n \ge 4$$

as shown in Figure 9.2. So, by the Squeeze Theorem it follows that

$$\lim_{n \to \infty} (-1)^n \frac{1}{n!} = 0.$$

**NOTE**    Example 5 suggests something about the rate at which $n!$ increases as $n \to \infty$. As Figure 9.2 suggests, both $1/2^n$ and $1/n!$ approach 0 as $n \to \infty$. Yet $1/n!$ approaches 0 so much faster than $1/2^n$ does that

$$\lim_{n \to \infty} \frac{1/n!}{1/2^n} = \lim_{n \to \infty} \frac{2^n}{n!} = 0.$$

In fact, it can be shown that for any fixed number $k$,

$$\lim_{n \to \infty} \frac{k^n}{n!} = 0.$$

This means that *the factorial function grows faster than any exponential function.*

In Example 5, the sequence $\{c_n\}$ has both positive and negative terms. For this sequence, it happens that the sequence of absolute values, $\{|c_n|\}$, also converges to 0. You can show this by the Squeeze Theorem using the inequality

$$0 \le \frac{1}{n!} \le \frac{1}{2^n}, \quad n \ge 4.$$

In such cases, it is often convenient to consider the sequence of absolute values—and then apply Theorem 9.4, which states that if the absolute value sequence converges to 0, the original signed sequence also converges to 0.

---

**THEOREM 9.4**    **Absolute Value Theorem**

For the sequence $\{a_n\}$, if

$$\lim_{n \to \infty} |a_n| = 0 \quad\quad \text{then} \quad\quad \lim_{n \to \infty} a_n = 0.$$

---

**Proof**   Consider the two sequences $\{|a_n|\}$ and $\{-|a_n|\}$. Because both of these sequences converge to 0 and

$$-|a_n| \le a_n \le |a_n|$$

you can use the Squeeze Theorem to conclude that $\{a_n\}$ converges to 0.     ▬▬▬

## Pattern Recognition for Sequences

Sometimes the terms of a sequence are generated by some rule that does not explicitly identify the $n$th term of the sequence. In such cases, you may be required to discover a *pattern* in the sequence and to describe the $n$th term. Once the $n$th term has been specified, you can investigate the convergence or divergence of the sequence.

### EXAMPLE 6   Finding the $n$th Term of a Sequence

Find a sequence $\{a_n\}$ whose first five terms are

$$\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}, \ldots$$

and then determine whether the particular sequence you have chosen converges or diverges.

**Solution**   First, note that the numerators are successive powers of 2, and the denominators form the sequence of positive odd integers. By comparing $a_n$ with $n$, you have the following pattern.

$$\frac{2^1}{1}, \frac{2^2}{3}, \frac{2^3}{5}, \frac{2^4}{7}, \frac{2^5}{9}, \ldots, \frac{2^n}{2n-1}$$

Using L'Hôpital's Rule to evaluate the limit of $f(x) = 2^x/(2x - 1)$, you obtain

$$\lim_{x \to \infty} \frac{2^x}{2x - 1} = \lim_{x \to \infty} \frac{2^x(\ln 2)}{2} = \infty \quad \Longrightarrow \quad \lim_{n \to \infty} \frac{2^n}{2n - 1} = \infty.$$

So, the sequence diverges.     ▬▬▬

Without a specific rule for generating the terms of a sequence or some knowledge of the context in which the terms of the sequence are obtained, it is not possible to determine the convergence or divergence of the sequence merely from its first several terms. For instance, although the first three terms of the following four sequences are identical, the first two sequences converge to 0, the third sequence converges to $\frac{1}{9}$, and the fourth sequence diverges.

$$\{a_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots \cdots, \frac{1}{2^n}, \cdots$$

$$\{b_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{15}, \cdots \cdots, \frac{6}{(n+1)(n^2-n+6)}, \cdots$$

$$\{c_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{7}{62}, \cdots \cdots, \frac{n^2-3n+3}{9n^2-25n+18}, \cdots$$

$$\{d_n\}: \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 0, \cdots \cdots, \frac{-n(n+1)(n-4)}{6(n^2+3n-2)}, \cdots$$

The process of determining an $n$th term from the pattern observed in the first several terms of a sequence is an example of *inductive reasoning*.

## EXAMPLE 7    Finding the $n$th Term of a Sequence

Determine an $n$th term for a sequence whose first five terms are

$$-\frac{2}{1}, \frac{8}{2}, -\frac{26}{6}, \frac{80}{24}, -\frac{242}{120}, \cdots$$

and then decide whether the sequence converges or diverges.

**Solution**    Note that the numerators are 1 less than $3^n$. So, you can reason that the numerators are given by the rule $3^n - 1$. Factoring the denominators produces

$$1 = 1$$
$$2 = 1 \cdot 2$$
$$6 = 1 \cdot 2 \cdot 3$$
$$24 = 1 \cdot 2 \cdot 3 \cdot 4$$
$$120 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots.$$

This suggests that the denominators are represented by $n!$. Finally, because the signs alternate, you can write the $n$th term as

$$a_n = (-1)^n \left( \frac{3^n - 1}{n!} \right).$$

From the discussion about the growth of $n!$, it follows that

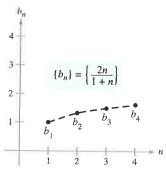$$\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \frac{3^n - 1}{n!} = 0.$$

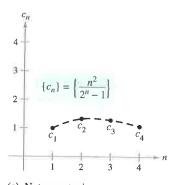Applying Theorem 9.4, you can conclude that

$$\lim_{n \to \infty} a_n = 0.$$

So, the sequence $\{a_n\}$ converges to 0.

## Monotonic Sequences and Bounded Sequences

So far you have determined the convergence of a sequence by finding its limit. Even if you cannot determine the limit of a particular sequence, it still may be useful to know whether the sequence converges. Theorem 9.5 provides a test for convergence of sequences without determining the limit. First, some preliminary definitions are given.



$\frac{a_n}{}$

$\{a_n\} = \{3 + (-1)^n\}$

**(a)** Not monotonic

---

> **Definition of a Monotonic Sequence**
>
> A sequence $\{a_n\}$ is **monotonic** if its terms are nondecreasing
> $$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$$
> or if its terms are nonincreasing
> $$a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge \cdots .$$

---

### EXAMPLE 8   Determining Whether a Sequence Is Monotonic

Determine whether each sequence having the given $n$th term is monotonic.

**a.** $a_n = 3 + (-1)^n$    **b.** $b_n = \dfrac{2n}{1 + n}$    **c.** $c_n = \dfrac{n^2}{2^n - 1}$

**Solution**

**a.** This sequence alternates between 2 and 4. So, it is not monotonic.

**b.** This sequence is monotonic because each successive term is larger than its predecessor. To see this, compare the terms $b_n$ and $b_{n+1}$. [Note that, because $n$ is positive, you can multiply each side of the inequality by $(1 + n)$ and $(2 + n)$ without reversing the inequality sign.]

$$b_n = \frac{2n}{1 + n} \overset{?}{<} \frac{2(n + 1)}{1 + (n + 1)} = b_{n+1}$$
$$2n(2 + n) \overset{?}{<} (1 + n)(2n + 2)$$
$$4n + 2n^2 \overset{?}{<} 2 + 4n + 2n^2$$
$$0 < 2$$

Starting with the final inequality, which is valid, you can reverse the steps to conclude that the original inequality is also valid.

**c.** This sequence is not monotonic, because the second term is larger than the first term, and larger than the third. (Note that if you drop the first term, the remaining sequence $c_2, c_3, c_4, \ldots$ is monotonic.)

Figure 9.3 graphically illustrates these three sequences.

$\{b_n\} = \left\{\dfrac{2n}{1 + n}\right\}$

**(b)** Monotonic

$\{c_n\} = \left\{\dfrac{n^2}{2^n - 1}\right\}$

**(c)** Not monotonic

**Figure 9.3**

**NOTE**   In Example 8(b), another way to see that the sequence is monotonic is to argue that the derivative of the corresponding differentiable function $f(x) = 2x/(1 + x)$ is positive for all $x$. This implies that $f$ is increasing, which in turn implies that $\{a_n\}$ is increasing.
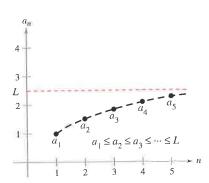
NOTE    All three sequences shown in Figure 9.3 are bounded. To see this, consider the following.

$$2 \leq a_n \leq 4$$

$$1 \leq b_n \leq 2$$

$$0 \leq c_n \leq \frac{4}{3}$$

**Definition of a Bounded Sequence**

1. A sequence $\{a_n\}$ is **bounded above** if there is a real number $M$ such that $a_n \leq M$ for all $n$. The number $M$ is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** if there is a real number $N$ such that $N \leq a_n$ for all $n$. The number $N$ is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** if it is bounded above and bounded below.

One important property of the real numbers is that they are **complete.** Informally, this means that there are no holes or gaps on the real number line. (The set of rational numbers does not have the completeness property.) The completeness axiom for real numbers can be used to conclude that if a sequence has an upper bound, it must have a **least upper bound** (an upper bound that is smaller than all other upper bounds for the sequence). For example, the least upper bound of the sequence $\{a_n\} = \{n/(n + 1)\}$,

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n + 1}, \ldots$$

is 1. The completeness axiom is used in the proof of Theorem 9.5.

**THEOREM 9.5    Bounded Monotonic Sequences**

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges.



Every bounded nondecreasing sequence converges.
**Figure 9.4**

**Proof**    Assume that the sequence is nondecreasing, as shown in Figure 9.4. For the sake of simplicity, also assume that each term in the sequence is positive. Because the sequence is bounded, there must exist an upper bound $M$ such that

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots \leq M.$$

From the completeness axiom, it follows that there is a least upper bound $L$ such that

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots \leq L.$$

For $\varepsilon > 0$, it follows that $L - \varepsilon < L$, and therefore $L - \varepsilon$ cannot be an upper bound for the sequence. Consequently, at least one term of $\{a_n\}$ is greater than $L - \varepsilon$. That is, $L - \varepsilon < a_N$ for some positive integer $N$. Because the terms of $\{a_n\}$ are nondecreasing, it follows that $a_N \leq a_n$ for $n > N$. You now know that $L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon$, for every $n > N$. It follows that $|a_n - L| < \varepsilon$ for $n > N$, which by definition means that $\{a_n\}$ converges to $L$. The proof for a nonincreasing sequence is similar.

**EXAMPLE 9    Bounded and Monotonic Sequences**

a. The sequence $\{a_n\} = \{1/n\}$ is both bounded and monotonic and so, by Theorem 9.5, must converge.
b. The divergent sequence $\{b_n\} = \{n^2/(n + 1)\}$ is monotonic, but not bounded. (It is bounded below.)
c. The divergent sequence $\{c_n\} = \{(-1)^n\}$ is bounded, but not monotonic.

**Exercises for Section 9.1**

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

**In Exercises 1–10, write the first five terms of the sequence.**

**1.** $a_n = 2^n$

**2.** $a_n = \dfrac{3^n}{n!}$

**3.** $a_n = \left(-\dfrac{1}{2}\right)^n$

**4.** $a_n = \left(-\dfrac{2}{3}\right)^n$

**5.** $a_n = \sin\dfrac{n\pi}{2}$

**6.** $a_n = \dfrac{2n}{n+3}$

**7.** $a_n = \dfrac{(-1)^{n(n+1)/2}}{n^2}$

**8.** $a_n = (-1)^{n+1}\left(\dfrac{2}{n}\right)$

**9.** $a_n = 5 - \dfrac{1}{n} + \dfrac{1}{n^2}$

**10.** $a_n = 10 + \dfrac{2}{n} + \dfrac{6}{n^2}$

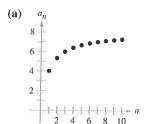**In Exercises 11–14, write the first five terms of the recursively defined sequence.**
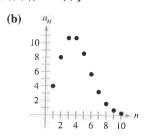
**11.** $a_1 = 3,\ a_{k+1} = 2(a_k - 1)$

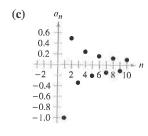**12.** $a_1 = 4,\ a_{k+1} = \left(\dfrac{k+1}{2}\right)a_k$
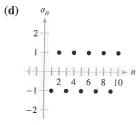
**13.** $a_1 = 32,\ a_{k+1} = \dfrac{1}{2}a_k$
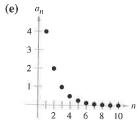
**14.** $a_1 = 6,\ a_{k+1} = \dfrac{1}{3}a_k^2$

**In Exercises 15–20, match the sequence with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**
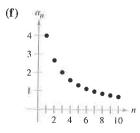
**(a)**

**(b)**

**(c)**

**(d)**

**(e)**

**(f)**

**15.** $a_n = \dfrac{8}{n+1}$

**16.** $a_n = \dfrac{8n}{n+1}$

**17.** $a_n = 4(0.5)^{n-1}$

**18.** $a_n = \dfrac{4^n}{n!}$

**19.** $a_n = (-1)^n$

**20.** $a_n = \dfrac{(-1)^n}{n}$

**In Exercises 21–24, use a graphing utility to graph the first 10 terms of the sequence.**

**21.** $a_n = \dfrac{2}{3}n$

**22.** $a_n = 2 - \dfrac{4}{n}$

**23.** $a_n = 16(-0.5)^{n-1}$

**24.** $a_n = \dfrac{2n}{n+1}$

**In Exercises 25–30, write the next two apparent terms of the sequence. Describe the pattern you used to find these terms.**

**25.** $2, 5, 8, 11, \ldots$

**26.** $\dfrac{7}{2}, 4, \dfrac{9}{2}, 5, \ldots$

**27.** $5, 10, 20, 40, \ldots$

**28.** $1, -\dfrac{1}{2}, \dfrac{1}{4}, -\dfrac{1}{8}, \ldots$

**29.** $3, -\dfrac{3}{2}, \dfrac{3}{4}, -\dfrac{3}{8}, \ldots$

**30.** $1, -\dfrac{3}{2}, \dfrac{9}{4}, -\dfrac{27}{8}, \ldots$

**In Exercises 31–36, simplify the ratio of factorials.**

**31.** $\dfrac{10!}{8!}$

**32.** $\dfrac{25!}{23!}$

**33.** $\dfrac{(n+1)!}{n!}$

**34.** $\dfrac{(n+2)!}{n!}$

**35.** $\dfrac{(2n-1)!}{(2n+1)!}$

**36.** $\dfrac{(2n+2)!}{(2n)!}$

**In Exercises 37–42, find the limit (if possible) of the sequence.**

**37.** $a_n = \dfrac{5n^2}{n^2+2}$

**38.** $a_n = 5 - \dfrac{1}{n^2}$

**39.** $a_n = \dfrac{2n}{\sqrt{n^2+1}}$

**40.** $a_n = \dfrac{5n}{\sqrt{n^2+4}}$

**41.** $a_n = \sin\dfrac{1}{n}$

**42.** $a_n = \cos\dfrac{2}{n}$

**In Exercises 43–46, use a graphing utility to graph the first 10 terms of the sequence. Use the graph to make an inference about the convergence or divergence of the sequence. Verify your inference analytically and, if the sequence converges, find its limit.**

**43.** $a_n = \dfrac{n+1}{n}$

**44.** $a_n = \dfrac{1}{n^{3/2}}$

**45.** $a_n = \cos\dfrac{n\pi}{2}$

**46.** $a_n = 3 - \dfrac{1}{2^n}$

**In Exercises 47–68, determine the convergence or divergence of the sequence with the given $n$th term. If the sequence converges, find its limit.**

**47.** $a_n = (-1)^n\left(\dfrac{n}{n+1}\right)$

**48.** $a_n = 1 + (-1)^n$

**49.** $a_n = \dfrac{3n^2 - n + 4}{2n^2 + 1}$

**50.** $a_n = \dfrac{\sqrt[3]{n}}{\sqrt[3]{n}+1}$

**51.** $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2n)^n}$

**52.** $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$

**53.** $a_n = \dfrac{1 + (-1)^n}{n}$

**54.** $a_n = \dfrac{1 + (-1)^n}{n^2}$

**55.** $a_n = \dfrac{\ln(n^3)}{2n}$

**56.** $a_n = \dfrac{\ln \sqrt{n}}{n}$

**57.** $a_n = \dfrac{3^n}{4^n}$

**58.** $a_n = (0.5)^n$

**59.** $a_n = \dfrac{(n+1)!}{n!}$

**60.** $a_n = \dfrac{(n-2)!}{n!}$

**61.** $a_n = \dfrac{n-1}{n} - \dfrac{n}{n-1}, \quad n \geq 2$

**62.** $a_n = \dfrac{n^2}{2n+1} - \dfrac{n^2}{2n-1}$

**63.** $a_n = \dfrac{n^p}{e^n}, \quad p > 0$

**64.** $a_n = n \sin \dfrac{1}{n}$

**65.** $a_n = \left(1 + \dfrac{k}{n}\right)^n$

**66.** $a_n = 2^{1/n}$

**67.** $a_n = \dfrac{\sin n}{n}$

**68.** $a_n = \dfrac{\cos \pi n}{n^2}$

**In Exercises 69–82, write an expression for the $n$th term of the sequence. (There is more than one correct answer.)**

**69.** $1, 4, 7, 10, \ldots$

**70.** $3, 7, 11, 15, \ldots$

**71.** $-1, 2, 7, 14, 23, \ldots$

**72.** $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \ldots$

**73.** $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

**74.** $2, -1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots$

**75.** $2, 1 + \frac{1}{2}, 1 + \frac{1}{3}, 1 + \frac{1}{4}, 1 + \frac{1}{5}, \ldots$

**76.** $1 + \frac{1}{2}, 1 + \frac{3}{4}, 1 + \frac{7}{8}, 1 + \frac{15}{16}, 1 + \frac{31}{32}, \ldots$

**77.** $\dfrac{1}{2 \cdot 3}, \dfrac{2}{3 \cdot 4}, \dfrac{3}{4 \cdot 5}, \dfrac{4}{5 \cdot 6}, \ldots$

**78.** $1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \ldots$

**79.** $1, -\dfrac{1}{1 \cdot 3}, \dfrac{1}{1 \cdot 3 \cdot 5}, -\dfrac{1}{1 \cdot 3 \cdot 5 \cdot 7}, \ldots$

**80.** $1, x, \dfrac{x^2}{2}, \dfrac{x^3}{6}, \dfrac{x^4}{24}, \dfrac{x^5}{120}, \ldots$

**81.** $2, 24, 720, 40{,}320, 3{,}628{,}800, \ldots$

**82.** $1, 6, 120, 5040, 362{,}880, \ldots$

**In Exercises 83–94, determine whether the sequence with the given $n$th term is monotonic. Discuss the boundedness of the sequence. Use a graphing utility to confirm your results.**

**83.** $a_n = 4 - \dfrac{1}{n}$

**84.** $a_n = \dfrac{3n}{n+2}$

**85.** $a_n = \dfrac{n}{2^{n+2}}$

**86.** $a_n = ne^{-n/2}$

**87.** $a_n = (-1)^n \left(\dfrac{1}{n}\right)$

**88.** $a_n = \left(-\dfrac{2}{3}\right)^n$

**89.** $a_n = \left(\dfrac{2}{3}\right)^n$

**90.** $a_n = \left(\dfrac{3}{2}\right)^n$

**91.** $a_n = \sin \dfrac{n\pi}{6}$

**92.** $a_n = \cos\left(\dfrac{n\pi}{2}\right)$

**93.** $a_n = \dfrac{\cos n}{n}$

**94.** $a_n = \dfrac{\sin \sqrt{n}}{n}$

**In Exercises 95–98, (a) use Theorem 9.5 to show that the sequence with the given $n$th term converges and (b) use a graphing utility to graph the first 10 terms of the sequence and find its limit.**

**95.** $a_n = 5 + \dfrac{1}{n}$

**96.** $a_n = 4 - \dfrac{3}{n}$

**97.** $a_n = \dfrac{1}{3}\left(1 - \dfrac{1}{3^n}\right)$

**98.** $a_n = 4 + \dfrac{1}{2^n}$

**99.** Let $\{a_n\}$ be an increasing sequence such that $2 \leq a_n \leq 4$. Explain why $\{a_n\}$ has a limit. What can you conclude about the limit?

**100.** Let $\{a_n\}$ be a monotonic sequence such that $a_n \leq 1$. Discuss the convergence of $\{a_n\}$. If $\{a_n\}$ converges, what can you conclude about its limit?

**101.** *Compound Interest*   Consider the sequence $\{A_n\}$ whose $n$th term is given by

$$A_n = P\left(1 + \dfrac{r}{12}\right)^n$$

where $P$ is the principal, $A_n$ is the account balance after $n$ months, and $r$ is the interest rate compounded annually.

(a) Is $\{A_n\}$ a convergent sequence? Explain.

(b) Find the first 10 terms of the sequence if $P = \$9000$ and $r = 0.055$.

**102.** *Compound Interest*   A deposit of $\$100$ is made at the beginning of each month in an account at an annual interest rate of $3\%$ compounded monthly. The balance in the account after $n$ months is $A_n = 100(401)(1.0025^n - 1)$.

(a) Compute the first six terms of the sequence $\{A_n\}$.

(b) Find the balance in the account after 5 years by computing the 60th term of the sequence.

(c) Find the balance in the account after 20 years by computing the 240th term of the sequence.
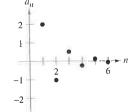
## Writing About Concepts

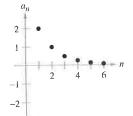**103.** In your own words, define each of the following.

(a) Sequence          (b) Convergence of a sequence

(c) Monotonic sequence   (d) Bounded sequence

**104.** The graphs of two sequences are shown in the figures. Which graph represents the sequence with alternating signs? Explain.

In Exercises 105–108, give an example of a sequence satisfying the condition or explain why no such sequence exists. (Examples are not unique.)

**105.** A monotonically increasing sequence that converges to 10

**106.** A monotonically increasing bounded sequence that does not converge

**107.** A sequence that converges to $\frac{3}{4}$

**108.** An unbounded sequence that converges to 100

**109.** *Government Expenditures* A government program that currently costs taxpayers \$2.5 billion per year is cut back by 20 percent per year.

(a) Write an expression for the amount budgeted for this program after $n$ years.

(b) Compute the budgets for the first 4 years.

(c) Determine the convergence or divergence of the sequence of reduced budgets. If the sequence converges, find its limit.

**110.** *Inflation* If the rate of inflation is $4\frac{1}{2}\%$ per year and the average price of a car is currently \$16,000, the average price after $n$ years is

$$P_n = \$16{,}000(1.045)^n.$$

Compute the average prices for the next 5 years.

**111.** *Modeling Data* The number $a_n$ of endangered and threatened species in the United States from 1996 through 2002 is shown in the table, where $n$ represents the year, with $n = 6$ corresponding to 1996. *(Source: U.S. Fish and Wildlife Service)*

| $n$ | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $a_n$ | 1053 | 1132 | 1194 | 1205 | 1244 | 1254 | 1263 |

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$a_n = bn^2 + cn + d, \qquad n = 6, 7, \ldots, 12$$

for the data. Use the graphing utility to plot the points and graph the model.

(b) Use the model to predict the number of endangered and threatened species in the year 2008.

**112.** *Modeling Data* The annual sales $a_n$ (in millions of dollars) for Avon Products, Inc. from 1993 through 2002 are given below as ordered pairs of the form $(n, a_n)$, where $n$ represents the year, with $n = 3$ corresponding to 1993. *(Source: 2002 Avon Products, Inc. Annual Report)*

(3, 3844), (4, 4267), (5, 4492), (6, 4814), (7, 5079), (8, 5213), (9, 5289), (10, 5682), (11, 5958), (12, 6171)

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$a_n = bn + c, \quad n = 3, 4, \ldots, 12$$

for the data. Graphically compare the points and the model.

(b) Use the model to predict sales in the year 2008.

**113.** *Comparing Exponential and Factorial Growth* Consider the sequence $a_n = 10^n/n!$.

(a) Find two consecutive terms that are equal in magnitude.

(b) Are the terms following those found in part (a) increasing or decreasing?

(c) In Section 8.7, Exercises 65–70, it was shown that for "large" values of the independent variable an exponential function increases more rapidly than a polynomial function. From the result in part (b), what inference can you make about the rate of growth of an exponential function versus a factorial function for "large" integer values of $n$?

**114.** Compute the first six terms of the sequence

$$\{a_n\} = \left\{ \left(1 + \frac{1}{n}\right)^n \right\}.$$

If the sequence converges, find its limit.

**115.** Compute the first six terms of the sequence $\{a_n\} = \left\{ \sqrt[n]{n} \right\}$. If the sequence converges, find its limit.

**116.** Prove that if $\{s_n\}$ converges to $L$ and $L > 0$, then there exists a number $N$ such that $s_n > 0$ for $n > N$.

*True or False?* In Exercises 117–120, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**117.** If $\{a_n\}$ converges to 3 and $\{b_n\}$ converges to 2, then $\{a_n + b_n\}$ converges to 5.

**118.** If $\{a_n\}$ converges, then $\lim_{n \to \infty} (a_n - a_{n+1}) = 0$.

**119.** If $n > 1$, then $n! = n(n - 1)!$.

**120.** If $\{a_n\}$ converges, then $\{a_n/n\}$ converges to 0.

**121.** *Fibonacci Sequence* In a study of the progeny of rabbits, Fibonacci (ca. 1170–ca. 1240) encountered the sequence now bearing his name. It is defined recursively by

$$a_{n+2} = a_n + a_{n+1}, \qquad \text{where} \quad a_1 = 1 \text{ and } a_2 = 1.$$

(a) Write the first 12 terms of the sequence.

(b) Write the first 10 terms of the sequence defined by

$$b_n = \frac{a_{n+1}}{a_n}, \quad n \geq 1.$$

(c) Using the definition in part (b), show that

$$b_n = 1 + \frac{1}{b_{n-1}}.$$

(d) The **golden ratio** $\rho$ can be defined by $\lim_{n \to \infty} b_n = \rho$. Show that $\rho = 1 + 1/\rho$ and solve this equation for $\rho$.

**122. *Conjecture*** Let $x_0 = 1$ and consider the sequence $x_n$ given by the formula

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}, \qquad n = 1, 2, \ldots .$$

Use a graphing utility to compute the first 10 terms of the sequence and make a conjecture about the limit of the sequence.

**123.** Consider the sequence

$$\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots$$

(a) Compute the first five terms of this sequence.

(b) Write a recursion formula for $a_n$, for $n \geq 2$.

(c) Find $\lim\limits_{n \to \infty} a_n$.

**124.** Consider the sequence

$$\sqrt{6}, \sqrt{6 + \sqrt{6}}, \sqrt{6 + \sqrt{6 + \sqrt{6}}}, \ldots$$

(a) Compute the first five terms of this sequence.

(b) Write a recursion formula for $a_n$, for $n \geq 2$.

(c) Find $\lim\limits_{n \to \infty} a_n$.

**125.** Consider the sequence $\{a_n\}$ where $a_1 = \sqrt{k}$, $a_{n+1} = \sqrt{k + a_n}$, and $k > 0$.

(a) Show that $\{a_n\}$ is increasing and bounded.

(b) Prove that $\lim\limits_{n \to \infty} a_n$ exists.

(c) Find $\lim\limits_{n \to \infty} a_n$.

**126. *Arithmetic-Geometric Mean*** Let $a_0 > b_0 > 0$. Let $a_1$ be the arithmetic mean of $a_0$ and $b_0$ and let $b_1$ be the geometric mean of $a_0$ and $b_0$.

$$a_1 = \frac{a_0 + b_0}{2} \qquad \text{Arithmetic mean}$$
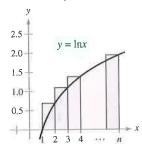
$$b_1 = \sqrt{a_0 b_0} \qquad \text{Geometric mean}$$

Now define the sequences $\{a_n\}$ and $\{b_n\}$ as follows.

$$a_n = \frac{a_{n-1} + b_{n-1}}{2} \qquad b_n = \sqrt{a_{n-1} b_{n-1}}$$

(a) Let $a_0 = 10$ and $b_0 = 3$. Write out the first five terms of $\{a_n\}$ and $\{b_n\}$. Compare the terms of $\{b_n\}$. Compare $a_n$ and $b_n$. What do you notice?

(b) Use induction to show that $a_n > a_{n+1} > b_{n+1} > b_n$, for $a_0 > b_0 > 0$.

(c) Explain why $\{a_n\}$ and $\{b_n\}$ are both convergent.

(d) Show that $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} b_n$.

**127.** (a) Let $f(x) = \sin x$ and $a_n = n \sin 1/n$. Show that

$$\lim_{n \to \infty} a_n - f'(0) = 1.$$

(b) Let $f(x)$ be differentiable on the interval $[0, 1]$ and $f(0) = 0$. Consider the sequence $\{a_n\}$, where $a_n = n f(1/n)$. Show that $\lim\limits_{n \to \infty} a_n = f'(0)$.

**128.** Consider the sequence $\{a_n\} = \{nr^n\}$. Decide whether $\{a_n\}$ converges for each value of $r$.

(a) $r = \frac{1}{2}$      (b) $r = 1$      (c) $r = \frac{3}{2}$

(d) For what values or $r$ does the sequence $\{nr^n\}$ converge?

**129.** (a) Show that $\int_1^n \ln x \, dx < \ln(n!)$ for $n \geq 2$.



(b) Draw a graph similar to the one above that shows

$$\ln(n!) < \int_1^{n+1} \ln x \, dx.$$

(c) Use the results of parts (a) and (b) to show that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}, \text{ for } n > 1.$$

(d) Use the Squeeze Theorem for Sequences and the result of part (c) to show that

$$\lim_{n \to \infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

(e) Test the result of part (d) for $n = 20, 50$, and $100$.

**130.** Consider the sequence $\{a_n\} = \left\{ \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} \dfrac{1}{1 + (k/n)} \right\}$.

(a) Write the first five terms of $\{a_n\}$.

(b) Show that $\lim\limits_{n \to \infty} a_n = \ln 2$ by interpreting $a_n$ as a Riemann sum of a definite integral.

**131.** Prove, using the definition of the limit of a sequence, that

$$\lim_{n \to \infty} \frac{1}{n^3} = 0.$$

**132.** Prove, using the definition of the limit of a sequence, that

$$\lim_{n \to \infty} r^n = 0 \text{ for } -1 < r < 1.$$

**133.** Complete the proof of Theorem 9.5.

## Putnam Exam Challenge

**134.** Let $\{x_n\}$, $n \geq 0$, be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1} x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$ . Prove that there exists a real number $a$ such that $x_{n+1} = a x_n - x_{n-1}$, for all $n \geq 1$.

**135.** Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and, for $n \geq 3$,

$$T_n = (n + 4)T_{n-1} - 4nT_{n-2} + (4n - 8)T_{n-3}.$$

The first 10 terms of the sequence are

$2, 3, 6, 14, 40, 152, 784, 5168, 40{,}576, 363{,}392.$

Find, with proof, a formula for $T_n$ of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

## Series and Convergence

- Understand the definition of a convergent infinite series.
- Use properties of infinite geometric series.
- Use the $n$th-Term Test for Divergence of an infinite series.

### Infinite Series

**INFINITE SERIES**

The study of infinite series was considered a novelty in the fourteenth century. Logician Richard Suiseth, whose nickname was Calculator, solved this problem.

*If throughout the first half of a given time interval a variation continues at a certain intensity, throughout the next quarter of the interval at double the intensity, throughout the following eighth at triple the intensity and so ad infinitum; then the average intensity for the whole interval will be the intensity of the variation during the second subinterval (or double the intensity).*

This is the same as saying that the sum of the infinite series

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} + \cdots$$

is 2.

One important application of infinite sequences is in representing "infinite summations." Informally, if $\{a_n\}$ is an infinite sequence, then

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots \qquad \text{Infinite series}$$

is an **infinite series** (or simply a **series**). The numbers $a_1$, $a_2$, $a_3$, are the **terms** of the series. For some series it is convenient to begin the index at $n = 0$ (or some other integer). As a typesetting convention, it is common to represent an infinite series as simply $\Sigma \, a_n$. In such cases, the starting value for the index must be taken from the context of the statement.

To find the sum of an infinite series, consider the following **sequence of partial sums.**

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

If this sequence of partial sums converges, the series is said to converge and has the sum indicated in the following definition.

---

### Definitions of Convergent and Divergent Series

For the infinite series $\displaystyle\sum_{n=1}^{\infty} a_n$, the **$n$th partial sum** is given by

$$S_n = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums $\{S_n\}$ converges to $S$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ **converges.** The limit $S$ is called the **sum of the series.**

$$S = a_1 + a_2 + \cdots + a_n + \cdots$$

If $\{S_n\}$ diverges, then the series **diverges.**

---

**STUDY TIP** As you study this chapter, you will see that there are two basic questions involving infinite series. Does a series converge or does it diverge? If a series converges, what is its sum? These questions are not always easy to answer, especially the second one.

### EXPLORATION

***Finding the Sum of an Infinite Series*** Find the sum of each infinite series. Explain your reasoning.

**a.** $0.1 + 0.01 + 0.001 + 0.0001 + \cdots$

**b.** $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \cdots$

**c.** $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

**d.** $\frac{15}{100} + \frac{15}{10,000} + \frac{15}{1,000,000} + \cdots$
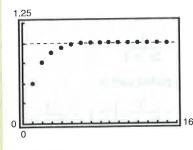
**Figure 9.5**

NOTE    You can geometrically determine the partial sums of the series in Example 1(a) using Figure 9.6.
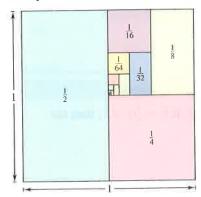


**Figure 9.6**

*FOR FURTHER INFORMATION*    To learn more about the partial sums of infinite series, see the article "Six Ways to Sum a Series" by Dan Kalman in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

**EXAMPLE 1    Convergent and Divergent Series**

**a.** The series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

has the following partial sums.

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$\vdots$$

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

Because

$$\lim_{n \to \infty} \frac{2^n - 1}{2^n} = 1$$

it follows that the series converges and its sum is 1.

**b.** The $n$th partial sum of the series

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = \left( 1 - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots$$

is given by

$$S_n = 1 - \frac{1}{n+1}.$$

Because the limit of $S_n$ is 1, the series converges and its sum is 1.

**c.** The series

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \cdots$$

diverges because $S_n = n$ and the sequence of partial sums diverges.

The series in Example 1(b) is a **telescoping series** of the form

$$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots. \qquad \text{Telescoping series}$$

Note that $b_2$ is canceled by the second term, $b_3$ is canceled by the third term, and so on. Because the $n$th partial sum of this series is

$$S_n = b_1 - b_{n+1}$$

it follows that a telescoping series will converge if and only if $b_n$ approaches a finite number as $n \to \infty$. Moreover, if the series converges, its sum is

$$S = b_1 - \lim_{n \to \infty} b_{n+1}.$$

### EXAMPLE 2    Writing a Series in Telescoping Form

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

**Solution**
Using partial fractions, you can write

$$a_n = \frac{2}{4n^2 - 1} = \frac{2}{(2n-1)(2n+1)} = \frac{1}{2n-1} - \frac{1}{2n+1}.$$

From this telescoping form, you can see that the $n$th partial sum is
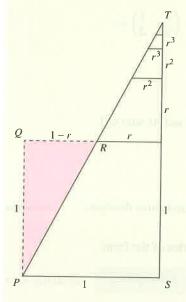
$$S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = 1 - \frac{1}{2n+1}.$$

So, the series converges and its sum is 1. That is,

$$\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{2n+1}\right) = 1.$$

## Geometric Series

The series given in Example 1(a) is a **geometric series.** In general, the series given by

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots, \quad a \neq 0 \qquad \text{Geometric series}$$

is a **geometric series** with ratio $r$.

---

**THEOREM 9.6    Convergence of a Geometric Series**

A geometric series with ratio $r$ diverges if $|r| \geq 1$. If $0 < |r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad 0 < |r| < 1.$$

---

**Proof**    It is easy to see that the series diverges if $r = \pm 1$. If $r \neq \pm 1$, then $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. Multiplication by $r$ yields

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n.$$

Subtracting the second equation from the first produces $S_n - rS_n = a - ar^n$. Therefore, $S_n(1-r) = a(1-r^n)$, and the $n$th partial sum is

$$S_n = \frac{a}{1-r}(1 - r^n).$$

If $0 < |r| < 1$, it follows that $r^n \to 0$ as $n \to \infty$, and you obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{a}{1-r}(1-r^n)\right] = \frac{a}{1-r}\left[\lim_{n\to\infty}(1-r^n)\right] = \frac{a}{1-r}$$

which means that the series *converges* and its sum is $a/(1-r)$. It is left to you to show that the series diverges if $|r| > 1$.

---

**EXAMPLE 3   Convergent and Divergent Geometric Series**

**a.** The geometric series

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = \sum_{n=0}^{\infty} 3\left(\frac{1}{2}\right)^n$$

$$= 3(1) + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + \cdots$$

has a ratio of $r = \frac{1}{2}$ with $a = 3$. Because $0 < |r| < 1$, the series converges and its sum is

$$S = \frac{a}{1 - r} = \frac{3}{1 - (1/2)} = 6.$$

**b.** The geometric series

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$$

has a ratio of $r = \frac{3}{2}$. Because $|r| \geq 1$, the series diverges.

The formula for the sum of a geometric series can be used to write a repeating decimal as the ratio of two integers, as demonstrated in the next example.

**EXAMPLE 4   A Geometric Series for a Repeating Decimal**

Use a geometric series to write $0.\overline{08}$ as the ratio of two integers.

**Solution**   For the repeating decimal $0.\overline{08}$, you can write

$$0.080808\ldots = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \frac{8}{10^8} + \cdots$$

$$= \sum_{n=0}^{\infty} \left(\frac{8}{10^2}\right)\left(\frac{1}{10^2}\right)^n.$$

For this series, you have $a = 8/10^2$ and $r = 1/10^2$. So,

$$0.080808\ldots = \frac{a}{1 - r} = \frac{8/10^2}{1 - (1/10^2)} = \frac{8}{99}.$$

Try dividing 8 by 99 on a calculator to see that it produces $0.\overline{08}$.

The convergence of a series is not affected by removal of a finite number of terms from the beginning of the series. For instance, the geometric series

$$\sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n \qquad \text{and} \qquad \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

both converge. Furthermore, because the sum of the second series is $a/(1 - r) = 2$, you can conclude that the sum of the first series is

$$S = 2 - \left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3\right]$$

$$= 2 - \frac{15}{8} = \frac{1}{8}.$$

**STUDY TIP**   As you study this chapter, it is important to distinguish between an infinite series and a sequence. A sequence is an ordered collection of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

whereas a series is an infinite sum of terms from a sequence

$$a_1 + a_2 + \cdots + a_n + \cdots.$$

The following properties are direct consequences of the corresponding properties of limits of sequences.

---

**THEOREM 9.7   Properties of Infinite Series**

If $\Sigma\, a_n = A$, $\Sigma\, b_n = B$, and $c$ is a real number, then the following series converge to the indicated sums.

1. $\displaystyle\sum_{n=1}^{\infty} c a_n = cA$

2. $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B$

3. $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = A - B$

---

## nth-Term Test for Divergence

The following theorem states that if a series converges, the limit of its $n$th term must be 0.

**NOTE**   Be sure you see that the converse of Theorem 9.8 is generally not true. That is, if the sequence $\{a_n\}$ converges to 0, then the series $\Sigma\, a_n$ may either converge or diverge.

---

**THEOREM 9.8   Limit of nth Term of a Convergent Series**

If $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\lim_{n \to \infty} a_n = 0$.

---

**Proof**   Assume that

$$\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = L.$$

Then, because $S_n = S_{n-1} + a_n$ and

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} S_{n-1} = L$$

it follows that

$$
\begin{aligned}
L = \lim_{n \to \infty} S_n &= \lim_{n \to \infty} (S_{n-1} + a_n) \\
&= \lim_{n \to \infty} S_{n-1} + \lim_{n \to \infty} a_n \\
&= L + \lim_{n \to \infty} a_n
\end{aligned}
$$

which implies that $\{a_n\}$ converges to 0.    ▬

The contrapositive of Theorem 9.8 provides a useful test for *divergence*. This **nth-Term Test for Divergence** states that if the limit of the $n$th term of a series does *not* converge to 0, the series must diverge.

---

**THEOREM 9.9   nth-Term Test for Divergence**

If $\displaystyle\lim_{n \to \infty} a_n \neq 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges.

---

## EXAMPLE 5  Using the *n*th-Term Test for Divergence

**a.** For the series $\displaystyle\sum_{n=0}^{\infty} 2^n$, you have

$$\lim_{n \to \infty} 2^n = \infty.$$

So, the limit of the *n*th term is not 0, and the series diverges.

**b.** For the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$, you have

$$\lim_{n \to \infty} \frac{n!}{2n! + 1} = \frac{1}{2}.$$

So, the limit of the *n*th term is not 0, and the series diverges.

**c.** For the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, you have

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

Because the limit of the *n*th term is 0, the *n*th-Term Test for Divergence does *not* apply and you can draw no conclusions about convergence or divergence. (In the next section, you will see that this particular series diverges.)

**STUDY TIP**  The series in Example 5(c) will play an important role in this chapter.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

You will see that this series diverges even though the *n*th term approaches 0 as *n* approaches $\infty$.

## EXAMPLE 6  Bouncing Ball Problem

A ball is dropped from a height of 6 feet and begins bouncing, as shown in Figure 9.7. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance traveled by the ball.

**Solution**  When the ball hits the ground for the first time, it has traveled a distance of $D_1 = 6$ feet. For subsequent bounces, let $D_i$ be the distance traveled up and down. For example, $D_2$ and $D_3$ are as follows.

$$D_2 = \underbrace{6\left(\tfrac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\tfrac{3}{4}\right)}_{\text{Down}} = 12\left(\tfrac{3}{4}\right)$$

$$D_3 = \underbrace{6\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)}_{\text{Up}} + \underbrace{6\left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)}_{\text{Down}} = 12\left(\tfrac{3}{4}\right)^2$$



The height of each bounce is three-fourths the height of the preceding bounce.
**Figure 9.7**

By continuing this process, it can be determined that the total vertical distance is

$$\begin{aligned}
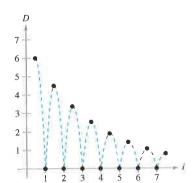D &= 6 + 12\left(\tfrac{3}{4}\right) + 12\left(\tfrac{3}{4}\right)^2 + 12\left(\tfrac{3}{4}\right)^3 + \cdots \\
&= 6 + 12 \sum_{n=0}^{\infty} \left(\tfrac{3}{4}\right)^{n+1} \\
&= 6 + 12\left(\tfrac{3}{4}\right) \sum_{n=0}^{\infty} \left(\tfrac{3}{4}\right)^n \\
&= 6 + 9\left(\frac{1}{1 - \frac{3}{4}}\right) \\
&= 6 + 9(4) \\
&= 42 \text{ feet.}
\end{aligned}$$

In Exercises 1–6, find the first five terms of the sequence of partial sums.

**1.** $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

**2.** $\frac{1}{2 \cdot 3} + \frac{2}{3 \cdot 4} + \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} + \frac{5}{6 \cdot 7} + \cdots$

**3.** $3 - \frac{9}{2} + \frac{27}{4} - \frac{81}{8} + \frac{243}{16} - \cdots$

**4.** $\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \cdots$

**5.** $\sum_{n=1}^{\infty} \frac{3}{2^{n-1}}$

**6.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$

In Exercises 7–16, verify that the infinite series diverges.

**7.** $\sum_{n=0}^{\infty} 3\left(\frac{3}{2}\right)^n$

**8.** $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n$

**9.** $\sum_{n=0}^{\infty} 1000(1.055)^n$

**10.** $\sum_{n=0}^{\infty} 2(-1.03)^n$

**11.** $\sum_{n=1}^{\infty} \frac{n}{n+1}$

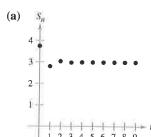**12.** $\sum_{n=1}^{\infty} \frac{n}{2n+3}$
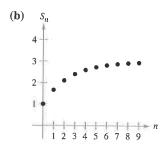
**13.** $\sum_{n=1}^{\infty} \frac{n^2}{n^2+1}$

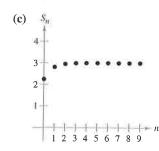**14.** $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$

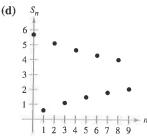**15.** $\sum_{n=1}^{\infty} \frac{2^n+1}{2^{n+1}}$
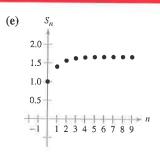
**16.** $\sum_{n=1}^{\infty} \frac{n!}{2^n}$

In Exercises 17–22, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).] Use the graph to estimate the sum of the series. Confirm your answer analytically.
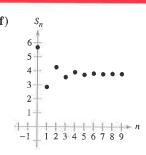
**(a)**

**(b)**

**(c)**

**(d)**

**(e)**

**(f)**

**17.** $\sum_{n=0}^{\infty} \frac{9}{4}\left(\frac{1}{4}\right)^n$

**18.** $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n$

**19.** $\sum_{n=0}^{\infty} \frac{15}{4}\left(-\frac{1}{4}\right)^n$

**20.** $\sum_{n=0}^{\infty} \frac{17}{3}\left(-\frac{8}{9}\right)^n$

**21.** $\sum_{n=0}^{\infty} \frac{17}{3}\left(-\frac{1}{2}\right)^n$

**22.** $\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n$

In Exercises 23–28, verify that the infinite series converges.

**23.** $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (Use partial fractions.)

**24.** $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ (Use partial fractions.)

**25.** $\sum_{n=0}^{\infty} 2\left(\frac{3}{4}\right)^n$

**26.** $\sum_{n=1}^{\infty} 2\left(-\frac{1}{2}\right)^n$

**27.** $\sum_{n=0}^{\infty} (0.9)^n = 1 + 0.9 + 0.81 + 0.729 + \cdots$

**28.** $\sum_{n=0}^{\infty} (-0.6)^n = 1 - 0.6 + 0.36 - 0.216 + \cdots$

*Numerical, Graphical, and Analytic Analysis*  In Exercises 29–34, (a) find the sum of the series, (b) use a graphing utility to find the indicated partial sum $S_n$ and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum, and (d) explain the relationship between the magnitudes of the terms of the series and the rate at which the sequence of partial sums approaches the sum of the series.

| $n$ | 5 | 10 | 20 | 50 | 100 |
|-----|---|----|----|----|-----|
| $S_n$ |  |  |  |  |  |

**29.** $\sum_{n=1}^{\infty} \frac{6}{n(n+3)}$

**30.** $\sum_{n=1}^{\infty} \frac{4}{n(n+4)}$

**31.** $\sum_{n=1}^{\infty} 2(0.9)^{n-1}$

**32.** $\sum_{n=1}^{\infty} 3(0.85)^{n-1}$

**33.** $\sum_{n=1}^{\infty} 10(0.25)^{n-1}$

**34.** $\sum_{n=1}^{\infty} 5\left(-\frac{1}{3}\right)^{n-1}$

In Exercises 35–50, find the sum of the convergent series.

**35.** $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$

**36.** $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$

**37.** $\sum\limits_{n=1}^{\infty} \dfrac{8}{(n+1)(n+2)}$

**38.** $\sum\limits_{n=1}^{\infty} \dfrac{1}{(2n+1)(2n+3)}$

**39.** $\sum\limits_{n=0}^{\infty} \left(\dfrac{1}{2}\right)^n$

**40.** $\sum\limits_{n=0}^{\infty} 6\left(\dfrac{4}{5}\right)^n$

**41.** $\sum\limits_{n=0}^{\infty} \left(-\dfrac{1}{2}\right)^n$

**42.** $\sum\limits_{n=0}^{\infty} 2\left(-\dfrac{2}{3}\right)^n$

**43.** $1 + 0.1 + 0.01 + 0.001 + \cdots$

**44.** $8 + 6 + \dfrac{9}{2} + \dfrac{27}{8} + \cdots$

**45.** $3 - 1 + \dfrac{1}{3} - \dfrac{1}{9} + \cdots$

**46.** $4 - 2 + 1 - \dfrac{1}{2} + \cdots$

**47.** $\sum\limits_{n=0}^{\infty} \left(\dfrac{1}{2^n} - \dfrac{1}{3^n}\right)$

**48.** $\sum\limits_{n=1}^{\infty} [(0.7)^n + (0.9)^n]$

**49.** $\sum\limits_{n=1}^{\infty} (\sin 1)^n$

**50.** $\sum\limits_{n=1}^{\infty} \dfrac{1}{9n^2 + 3n - 2}$

In Exercises 51–56, (a) write the repeating decimal as a geometric series and (b) write its sum as the ratio of two integers.

**51.** $0.\overline{4}$

**52.** $0.\overline{9}$

**53.** $0.\overline{81}$

**54.** $0.\overline{01}$

**55.** $0.0\overline{75}$

**56.** $0.2\overline{15}$

In Exercises 57–72, determine the convergence or divergence of the series.

**57.** $\sum\limits_{n=1}^{\infty} \dfrac{n+10}{10n+1}$

**58.** $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{2n-1}$

**59.** $\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{n} - \dfrac{1}{n+2}\right)$

**60.** $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+3)}$

**61.** $\sum\limits_{n=1}^{\infty} \dfrac{3n-1}{2n+1}$

**62.** $\sum\limits_{n=1}^{\infty} \dfrac{3^n}{n^3}$

**63.** $\sum\limits_{n=0}^{\infty} \dfrac{4}{2^n}$

**64.** $\sum\limits_{n=0}^{\infty} \dfrac{1}{4^n}$

**65.** $\sum\limits_{n=0}^{\infty} (1.075)^n$

**66.** $\sum\limits_{n=1}^{\infty} \dfrac{2^n}{100}$

**67.** $\sum\limits_{n=2}^{\infty} \dfrac{n}{\ln n}$

**68.** $\sum\limits_{n=1}^{\infty} \ln \dfrac{1}{n}$

**69.** $\sum\limits_{n=1}^{\infty} \left(1 + \dfrac{k}{n}\right)^n$

**70.** $\sum\limits_{n=1}^{\infty} e^{-n}$

**71.** $\sum\limits_{n=1}^{\infty} \arctan n$

**72.** $\sum\limits_{n=1}^{\infty} \ln\left(\dfrac{n+1}{n}\right)$

## Writing About Concepts

**73.** State the definitions of convergent and divergent series.

**74.** Describe the difference between $\lim\limits_{n \to \infty} a_n = 5$ and

$$\sum\limits_{n=1}^{\infty} a_n = 5.$$

**75.** Define a geometric series, state when it converges, and give the formula for the sum of a convergent geometric series.

## Writing About Concepts *(continued)*

**76.** State the $n$th-Term Test for Divergence.

**77.** Let $a_n = \dfrac{n+1}{n}$. Discuss the convergence of $\{a_n\}$ and

$$\sum\limits_{n=1}^{\infty} a_n.$$

**78.** Explain any differences among the following series.

(a) $\sum\limits_{n=1}^{\infty} a_n$    (b) $\sum\limits_{k=1}^{\infty} a_k$    (c) $\sum\limits_{n=1}^{\infty} a_k$

In Exercises 79–86, find all values of $x$ for which the series converges. For these values of $x$, write the sum of the series as a function of $x$.

**79.** $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{2^n}$

**80.** $\sum\limits_{n=1}^{\infty} (3x)^n$

**81.** $\sum\limits_{n=1}^{\infty} (x-1)^n$

**82.** $\sum\limits_{n=0}^{\infty} 4\left(\dfrac{x-3}{4}\right)^n$

**83.** $\sum\limits_{n=0}^{\infty} (-1)^n x^n$

**84.** $\sum\limits_{n=0}^{\infty} (-1)^n x^{2n}$

**85.** $\sum\limits_{n=0}^{\infty} \left(\dfrac{1}{x}\right)^n$

**86.** $\sum\limits_{n=1}^{\infty} \left(\dfrac{x^2}{x^2+4}\right)^n$

**87.** (a) You delete a finite number of terms from a divergent series. Will the new series still diverge? Explain your reasoning.

(b) You add a finite number of terms to a convergent series. Will the new series still converge? Explain your reasoning.

**88.** *Think About It*    Consider the formula

$$\dfrac{1}{x-1} = 1 + x + x^2 + x^3 + \cdots.$$

Given $x = -1$ and $x = 2$, can you conclude that either of the following statements is true? Explain your reasoning.

(a) $\dfrac{1}{2} = 1 - 1 + 1 - 1 + \cdots$

(b) $-1 = 1 + 2 + 4 + 8 + \cdots$

In Exercises 89 and 90, (a) find the common ratio of the geometric series, (b) write the function that gives the sum of the series, and (c) use a graphing utility to graph the function and the partial sums $S_3$ and $S_5$. What do you notice?

**89.** $1 + x + x^2 + x^3 + \cdots$

**90.** $1 - \dfrac{x}{2} + \dfrac{x^2}{4} - \dfrac{x^3}{8} + \cdots$

In Exercises 91 and 92, use a graphing utility to graph the function. Identify the horizontal asymptote of the graph and determine its relationship to the sum of the series.

| *Function* | *Series* |
|---|---|
| **91.** $f(x) = 3\left[\dfrac{1-(0.5)^x}{1-0.5}\right]$ | $\sum\limits_{n=0}^{\infty} 3\left(\dfrac{1}{2}\right)^n$ |
| **92.** $f(x) = 2\left[\dfrac{1-(0.8)^x}{1-0.8}\right]$ | $\sum\limits_{n=0}^{\infty} 2\left(\dfrac{4}{5}\right)^n$ |

*Writing* **In Exercises 93 and 94, use a graphing utility to determine the first term that is less than 0.0001 in each of the convergent series. Note that the answers are very different. Explain how this will affect the rate at which the series converges.**

**93.** $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, $\sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n$  **94.** $\sum_{n=1}^{\infty} \frac{1}{2^n}$, $\sum_{n=1}^{\infty} (0.01)^n$

**95. Marketing** An electronic games manufacturer producing a new product estimates the annual sales to be 8000 units. Each year 10% of the units that have been sold will become inoperative. So, 8000 units will be in use after 1 year, $[8000 + 0.9(8000)]$ units will be in use after 2 years, and so on. How many units will be in use after $n$ years?

**96. Depreciation** A company buys a machine for \$225,000 that depreciates at a rate of 30% per year. Find a formula for the value of the machine after $n$ years. What is its value after 5 years?

**97. Multiplier Effect** The annual spending by tourists in a resort city is \$100 million. Approximately 75% of that revenue is again spent in the resort city, and of that amount approximately 75% is again spent in the same city, and so on. Write the geometric series that gives the total amount of spending generated by the \$100 million and find the sum of the series.

**98. Multiplier Effect** Repeat Exercise 97 if the percent of the revenue that is spent again in the city decreases to 60%.

**99. Distance** A ball is dropped from a height of 16 feet. Each time it drops $h$ feet, it rebounds $0.81h$ feet. Find the total distance traveled by the ball.

**100. Time** The ball in Exercise 99 takes the following times for each fall.

$s_1 = -16t^2 + 16,$       $s_1 = 0$ if $t = 1$

$s_2 = -16t^2 + 16(0.81),$       $s_2 = 0$ if $t = 0.9$

$s_3 = -16t^2 + 16(0.81)^2,$       $s_3 = 0$ if $t = (0.9)^2$

$s_4 = -16t^2 + 16(0.81)^3,$       $s_4 = 0$ if $t = (0.9)^3$

$\vdots$       $\vdots$

$s_n = -16t^2 + 16(0.81)^{n-1},$       $s_n = 0$ if $t = (0.9)^{n-1}$

Beginning with $s_2$, the ball takes the same amount of time to bounce up as it does to fall, and so the total time elapsed before it comes to rest is given by

$t = 1 + 2\sum_{n=1}^{\infty} (0.9)^n.$

Find this total time.

*Probability* **In Exercises 101 and 102, the random variable $n$ represents the number of units of a product sold per day in a store. The probability distribution of $n$ is given by $P(n)$. Find the probability that two units are sold in a given day $[P(2)]$ and show that $P(1) + P(2) + P(3) + \cdots = 1$.**

**101.** $P(n) = \frac{1}{2}\left(\frac{1}{2}\right)^n$       **102.** $P(n) = \frac{1}{3}\left(\frac{2}{3}\right)^n$

**103. Probability** A fair coin is tossed repeatedly. The probability that the first head occurs on the $n$th toss is given by $P(n) = \left(\frac{1}{2}\right)^n$, where $n \geq 1$.

(a) Show that $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 1.$

(b) The expected number of tosses required until the first head occurs in the experiment is given by

$\sum_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n.$

Is this series geometric?

(c) Use a computer algebra system to find the sum in part (b).

**104. Probability** In an experiment, three people toss a fair coin one at a time until one of them tosses a head. Determine, for each person, the probability that he or she tosses the first head. Verify that the sum of the three probabilities is 1.

**105. Area** The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the triangles outside the second square are shaded (see figure). Determine the area of the shaded regions (a) if this process is continued five more times and (b) if this pattern of shading is continued infinitely.
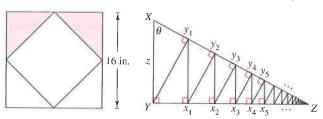


**Figure for 105**              **Figure for 106**

**106. Length** A right triangle $XYZ$ is shown above where $|XY| = z$ and $\angle X = \theta$. Line segments are continually drawn to be perpendicular to the triangle, as shown in the figure.

(a) Find the total length of the perpendicular line segments $|Yy_1| + |x_1y_1| + |x_1y_2| + \cdots$ in terms of $z$ and $\theta$.

(b) If $z = 1$ and $\theta = \pi/6$, find the total length of the perpendicular line segments.

**In Exercises 107–110, use the formula for the $n$th partial sum of a geometric series**

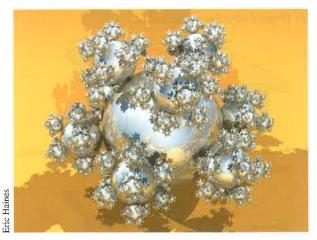$$\sum_{i=0}^{n-1} ar^i = \frac{a(1-r^n)}{1-r}.$$

**107. Present Value** The winner of a \$1,000,000 sweepstakes will be paid \$50,000 per year for 20 years. The money earns 6% interest per year. The present value of the winnings is

$$\sum_{n=1}^{20} 50,000\left(\frac{1}{1.06}\right)^n.$$

Compute the present value and interpret its meaning.

**108.** *Sphereflake*   A sphereflake shown below is a computer-generated fractal that was created by Eric Haines. The radius of the large sphere is 1. To the large sphere, nine spheres of radius $\frac{1}{3}$ are attached. To each of these, nine spheres of radius $\frac{1}{9}$ are attached. This process is continued infinitely. Prove that the sphereflake has an infinite surface area.


Eric Haines

**109.** *Salary*   You go to work at a company that pays $0.01 for the first day, $0.02 for the second day, $0.04 for the third day, and so on. If the daily wage keeps doubling, what would your total income be for working (a) 29 days, (b) 30 days, and (c) 31 days?

**110.** *Annuities*   When an employee receives a paycheck at the end of each month, $P$ dollars is invested in a retirement account. These deposits are made each month for $t$ years and the account earns interest at the annual percentage rate $r$. If the interest is compounded monthly, the amount $A$ in the account at the end of $t$ years is

$$A = P + P\left(1 + \frac{r}{12}\right) + \cdots + P\left(1 + \frac{r}{12}\right)^{12t-1}$$

$$= P\left(\frac{12}{r}\right)\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right].$$

If the interest is compounded continuously, the amount $A$ in the account after $t$ years is

$$A = P + Pe^{r/12} + Pe^{2r/12} + Pe^{(12t-1)r/12}$$

$$= \frac{P(e^{rt} - 1)}{e^{r/12} - 1}.$$

Verify the formulas for the sums given above.

*Annuities*   **In Exercises 111–114, consider making monthly deposits of $P$ dollars in a savings account at an annual interest rate $r$. Use the results of Exercise 110 to find the balance $A$ after $t$ years if the interest is compounded (a) monthly and (b) continuously.**

**111.** $P = \$50, \quad r = 3\%, \quad t = 20$ years

**112.** $P = \$75, \quad r = 5\%, \quad t = 25$ years

**113.** $P = \$100, \quad r = 4\%, \quad t = 40$ years

**114.** $P = \$20, \quad r = 6\%, \quad t = 50$ years

**115.** *Modeling Data*   The annual sales $a_n$ (in millions of dollars) for Avon Products, Inc. from 1993 through 2002 are given below as ordered pairs of the form $(n, a_n)$, where $n$ represents the year, with $n = 3$ corresponding to 1993.   *(Source: 2002 Avon Products, Inc. Annual Report)*

(3, 3844), (4, 4267), (5, 4492), (6, 4814), (7, 5079), (8, 5213), (9, 5289), (10, 5682), (11, 5958), (12, 6171)

(a) Use the regression capabilities of a graphing utility to find a model of the form

$$a_n = ce^{kn}, \quad n = 3, 4, 5, \ldots, 12$$

for the data. Graphically compare the points and the model.

(b) Use the data to find the total sales for the 10-year period.

(c) Approximate the total sales for the 10-year period using the formula for the sum of a geometric series. Compare the result with that in part (b).

**116.** *Salary*   You accept a job that pays a salary of $40,000 for the first year. During the next 39 years you receive a 4% raise each year. What would be your total compensation over the 40-year period?

*True or False?*   **In Exercises 117–122, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

**117.** If $\lim\limits_{n\to\infty} a_n = 0$, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

**118.** If $\sum\limits_{n=1}^{\infty} a_n = L$, then $\sum\limits_{n=0}^{\infty} a_n = L + a_0$.

**119.** If $|r| < 1$, then $\sum\limits_{n=1}^{\infty} ar^n = \frac{a}{(1-r)}$.

**120.** The series $\sum\limits_{n=1}^{\infty} \frac{n}{1000(n+1)}$ diverges.

**121.** $0.75 = 0.749999\ldots.$

**122.** Every decimal with a repeating pattern of digits is a rational number.

**123.** Show that the series $\sum\limits_{n=1}^{\infty} a_n$ can be written in the telescoping form

$$\sum_{n=1}^{\infty} [(c - S_{n-1}) - (c - S_n)]$$

where $S_0 = 0$ and $S_n$ is the $n$th partial sum.
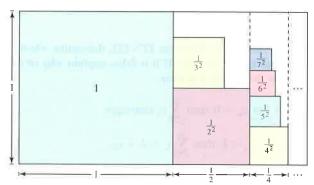
**124.** Let $\sum a_n$ be a convergent series, and let

$$R_N = a_{N+1} + a_{N+2} + \cdots$$

be the remainder of the series after the first $N$ terms. Prove that $\lim\limits_{N\to\infty} R_N = 0$.

**125.** Find two divergent series $\sum a_n$ and $\sum b_n$ such that $\sum(a_n + b_n)$ converges.

**126.** Given two infinite series $\sum a_n$ and $\sum b_n$ such that $\sum a_n$ converges and $\sum b_n$ diverges, prove that $\sum(a_n + b_n)$ diverges.

**127.** Suppose that $\sum a_n$ diverges and $c$ is a nonzero constant. Prove that $\sum ca_n$ diverges.

**128.** If $\sum_{n=1}^{\infty} a_n$ converges where $a_n$ is nonzero, show that $\sum_{n=1}^{\infty} \dfrac{1}{a_n}$ diverges.

**129.** The Fibonacci sequence is defined recursively by $a_{n+2} = a_n + a_{n+1}$, where $a_1 = 1$ and $a_2 = 1$.

(a) Show that $\dfrac{1}{a_{n+1} \, a_{n+3}} = \dfrac{1}{a_{n+1} \, a_{n+2}} - \dfrac{1}{a_{n+2} \, a_{n+3}}$.

(b) Show that $\sum_{n=0}^{\infty} \dfrac{1}{a_{n+1} \, a_{n+3}} = 1$.

**130.** Find the values of $x$ for which the infinite series

$$1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + x^6 + \cdots$$

converges. What is the sum when the series converges?

**131.** Prove that $\dfrac{1}{r} + \dfrac{1}{r^2} + \dfrac{1}{r^3} + \cdots = \dfrac{1}{r-1}$, for $|r| > 1$.

**132.** *Writing* The figure below represents an informal way of showing that $\sum_{n=1}^{\infty} \dfrac{1}{n^2} < 2$. Explain how the figure implies this conclusion.



**FOR FURTHER INFORMATION** For more on this exercise, see the article "Convergence with Pictures" by P.J. Rippon in *American Mathematical Monthly.*

**133.** *Writing* Read the article "The Exponential-Decay Law Applied to Medical Dosages" by Gerald M. Armstrong and Calvin P. Midgley in *Mathematics Teacher.* (To view this article, go to the website *www.matharticles.com.*) Then write a paragraph on how a geometric sequence can be used to find the total amount of a drug that remains in a patient's system after $n$ equal doses have been administered (at equal time intervals).

## Putnam Exam Challenge

**134.** Write $\sum_{k=1}^{\infty} \dfrac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$ as a rational number.

**135.** Let $f(n)$ be the sum of the first $n$ terms of the sequence $0, 1, 1, 2, 2, 3, 3, 4, \ldots$, where the $n$th term is given by

$$a_n = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd} \end{cases}.$$
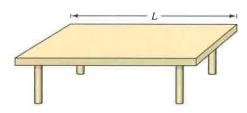
Show that if $x$ and $y$ are positive integers and $x > y$ then $xy = f(x+y) - f(x-y)$.
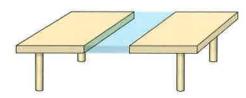
## Section Project: Cantor's Disappearing Table

The following procedure shows how to make a table disappear by removing only half of the table!
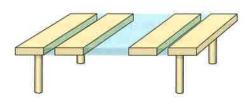
(a) Original table has a length of $L$.
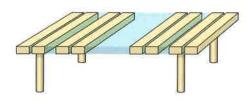


(b) Remove $\frac{1}{4}$ of the table centered at the midpoint. Each remaining piece has a length that is less than $\frac{1}{2}L$.



(c) Remove $\frac{1}{8}$ of the table by taking sections of length $\frac{1}{16}L$ from the centers of each of the two remaining pieces. Now, you have removed $\frac{1}{4} + \frac{1}{8}$ of the table. Each remaining piece has a length that is less than $\frac{1}{4}L$.



(d) Remove $\frac{1}{16}$ of the table by taking sections of length $\frac{1}{64}L$ from the centers of each of the four remaining pieces. Now, you have removed $\frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ of the table. Each remaining piece has a length that is less than $\frac{1}{8}L$.



Will continuing this process cause the table to disappear, even though you have only removed half of the table? Why?

**FOR FURTHER INFORMATION** Read the article "Cantor's Disappearing Table" by Larry E. Knop in *The College Mathematics Journal.* To view this article, go to the website *www.matharticles.com.*

**Section 9.3**

# The Integral Test and $p$-Series

- Use the Integral Test to determine whether an infinite series converges or diverges.
- Use properties of $p$-series and harmonic series.

## The Integral Test

In this and the following section, you will study several convergence tests that apply to series with *positive* terms.

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**THEOREM 9.10    The Integral Test**

If $f$ is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \int_1^{\infty} f(x)\, dx$$

either both converge or both diverge.

---

**Proof**   Begin by partitioning the interval $[1, n]$ into $n - 1$ unit intervals, as shown in Figure 9.8. The total areas of the inscribed rectangles and the circumscribed rectangles are as follows.

$$\sum_{i=2}^{n} f(i) = f(2) + f(3) + \cdots + f(n) \qquad \text{Inscribed area}$$

$$\sum_{i=1}^{n-1} f(i) = f(1) + f(2) + \cdots + f(n-1) \qquad \text{Circumscribed area}$$

The exact area under the graph of $f$ from $x = 1$ to $x = n$ lies between the inscribed and circumscribed areas.

$$\sum_{i=2}^{n} f(i) \leq \int_1^n f(x)\, dx \leq \sum_{i=1}^{n-1} f(i)$$

Using the $n$th partial sum, $S_n = f(1) + f(2) + \cdots + f(n)$, you can write this inequality as
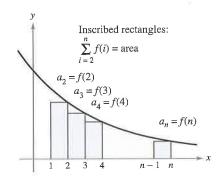
$$S_n - f(1) \leq \int_1^n f(x)\, dx \leq S_{n-1}.$$

Now, assuming that $\int_1^{\infty} f(x)\, dx$ converges to $L$, it follows that for $n \geq 1$

$$S_n - f(1) \leq L \quad \Longrightarrow \quad S_n \leq L + f(1).$$

Consequently, $\{S_n\}$ is bounded and monotonic, and by Theorem 9.5 it converges. So, $\sum a_n$ converges. For the other direction of the proof, assume that the improper integral diverges. Then $\int_1^n f(x)\, dx$ approaches infinity as $n \to \infty$, and the inequality $S_{n-1} \geq \int_1^n f(x)\, dx$ implies that $\{S_n\}$ diverges. So, $\sum a_n$ diverges. ▬▬▬

**NOTE**   Remember that the convergence or divergence of $\sum a_n$ is not affected by deleting the first $N$ terms. Similarly, if the conditions for the Integral Test are satisfied for all $x \geq N > 1$, you can simply use the integral $\int_N^{\infty} f(x)\, dx$ to test for convergence or divergence. (This is illustrated in Example 4.)



Inscribed rectangles:
$$\sum_{i=2}^{n} f(i) = \text{area}$$
$a_2 = f(2)$
$a_3 = f(3)$
$a_4 = f(4)$
$a_n = f(n)$

Circumscribed rectangles:
$$\sum_{i=1}^{n-1} f(i) = \text{area}$$
$a_1 = f(1)$
$a_2 = f(2)$
$a_3 = f(3)$
$a_{n-1} = f(n-1)$

**Figure 9.8**

**ion 9.4**

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

the graphs of the first 10
s of the sequence of par-

$$\frac{6}{n\sqrt{n^2 + 0.5}}$$

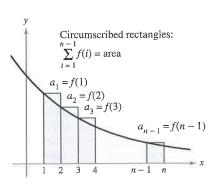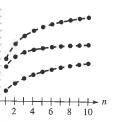converge or diverge?

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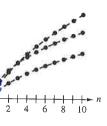ries?

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**In Exercises 3–14, use the Direct Comparison Test to determine the convergence or divergence of the series.**

**3.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$

**4.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2 + 2}$

**5.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n - 1}$

**6.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} - 1}$

**7.** $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n + 1}$

**8.** $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{4^n + 5}$

**9.** $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n + 1}$

**10.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 1}}$

**11.** $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$

**12.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n} - 1}$

**13.** $\displaystyle\sum_{n=0}^{\infty} e^{-n^2}$

**14.** $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{3^n - 1}$

**In Exercises 15–28, use the Limit Comparison Test to determine the convergence or divergence of the series.**

**15.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

**16.** $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n - 5}$

**17.** $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$

**18.** $\displaystyle\sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2 - 4}}$

**19.** $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 - 1}{3n^5 + 2n + 1}$

**20.** $\displaystyle\sum_{n=1}^{\infty} \frac{5n - 3}{n^2 - 2n + 5}$

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{n + 3}{n(n + 2)}$

**22.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n^2 + 1)}$

**23.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2 + 1}}$

**24.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n + 1)2^{n-1}}$

**25.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k + 1}, \quad k > 2$

**26.** $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n + \sqrt{n^2 + 4}}$

**27.** $\displaystyle\sum_{n=1}^{\infty} \sin \frac{1}{n}$

**28.** $\displaystyle\sum_{n=1}^{\infty} \tan \frac{1}{n}$

**In Exercises 29–36, test for convergence or divergence, using each test at least once. Identify which test was used.**

**(a)** $n$th-Term Test

**(b)** Geometric Series Test

**(c)** $p$-Series Test

**(d)** Telescoping Series Test

**(e)** Integral Test

**(f)** Direct Comparison Test

**(g)** Limit Comparison Test

**29.** $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$

**30.** $\displaystyle\sum_{n=0}^{\infty} 5\left(-\frac{1}{5}\right)^n$

**31.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + 2}$

**32.** $\displaystyle\sum_{n=4}^{\infty} \frac{1}{3n^2 - 2n - 15}$

**33.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n + 3}$

**34.** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n + 1} - \frac{1}{n + 2}\right)$

**35.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$

**36.** $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n + 3)}$

·ges.

c series. If its terms were
·ct it to diverge. However,
to expect. The function
o determine whether $f$ is
id its derivative.

conditions for the Integral

or instance, the sum of the first
irst 100 terms is just slightly
proximately 3.015021704. You
·es so very slowly.

. $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$

. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$

tegral Test to determine the
series, where $k$ is a positive

. $\displaystyle\sum_{n=1}^{\infty} n^k e^{-n}$

**37.** Use the Limit Comparison Test with the harmonic series to show that the series $\Sigma\, a_n$ (where $0 < a_n < a_{n-1}$) diverges if $\lim_{n\to\infty} na_n$ is finite and nonzero.

**38.** Prove that, if $P(n)$ and $Q(n)$ are polynomials of degree $j$ and $k$, respectively, then the series

$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$$

converges if $j < k - 1$ and diverges if $j \geq k - 1$.

**In Exercises 39–42, use the polynomial test given in Exercise 38 to determine whether the series converges or diverges.**

**39.** $\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \frac{4}{17} + \frac{5}{26} + \cdots$

**40.** $\frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{35} + \cdots$

**41.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3 + 1}$    **42.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$

**In Exercises 43 and 44, use the divergence test given in Exercise 37 to show that the series diverges.**

**43.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{5n^4 + 3}$    **44.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

**In Exercises 45–48, determine the convergence or divergence of the series.**

**45.** $\frac{1}{200} + \frac{1}{400} + \frac{1}{600} + \frac{1}{800} + \cdots$

**46.** $\frac{1}{200} + \frac{1}{210} + \frac{1}{220} + \frac{1}{230} + \cdots$

**47.** $\frac{1}{201} + \frac{1}{204} + \frac{1}{209} + \frac{1}{216} + \cdots$

**48.** $\frac{1}{201} + \frac{1}{208} + \frac{1}{227} + \frac{1}{264} + \cdots$

## Writing About Concepts

**49.** Review the results of Exercises 45–48. Explain why careful analysis is required to determine the convergence or divergence of a series and why only considering the magnitudes of the terms of a series could be misleading.

**50.** State the Direct Comparison Test and give an example of its use.

**51.** State the Limit Comparison Test and give an example of its use.

**52.** It appears that the terms of the series

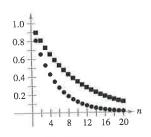$$\frac{1}{1000} + \frac{1}{1001} + \frac{1}{1002} + \frac{1}{1003} + \cdots$$

are less than the corresponding terms of the convergent series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots.$$

If the statement above is correct, the first series converges. Is this correct? Why or why not? Make a statement about how the divergence or convergence of a series is affected by inclusion or exclusion of the first finite number of terms.

## Writing About Concepts (continued)

**53.** The figure shows the first 20 terms of the convergent series $\displaystyle\sum_{n=1}^{\infty} a_n$ and the first 20 terms of the series $\displaystyle\sum_{n=1}^{\infty} a_n^2$. Identify the two series and explain your reasoning in making the selection.



**54.** Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

(a) Verify that the series converges.

(b) Use a graphing utility to complete the table.

| $n$ | 5 | 10 | 20 | 50 | 100 |
|-----|---|----|----|----|-----|
| $S_n$ | | | | | |

(c) The sum of the series is $\pi^2/8$. Find the sum of the series

$$\sum_{n=3}^{\infty} \frac{1}{(2n-1)^2}.$$

(d) Use a graphing utility to find the sum of the series

$$\sum_{n=10}^{\infty} \frac{1}{(2n-1)^2}.$$

*True or False?*  **In Exercises 55–60, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

**55.** If $0 < a_n \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

**56.** If $0 < a_{n+10} \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

**57.** If $a_n + b_n \leq c_n$ and $\displaystyle\sum_{n=1}^{\infty} c_n$ converges, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ both converge. (Assume that the terms of all three series are positive.)

**58.** If $a_n \leq b_n + c_n$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then the series $\displaystyle\sum_{n=1}^{\infty} b_n$ and $\displaystyle\sum_{n=1}^{\infty} c_n$ both diverge. (Assume that the terms of all three series are positive.)

**59.** If $0 < a_n \leq b_n$ and $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges.

### EXAMPLE 7 Rearrangement of a Series

The alternating harmonic series converges to ln 2. That is,

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2. \qquad \text{(See Exercise 49, Section 9.10.)}$$

Rearrange the series to produce a different sum.

**Solution**  Consider the following rearrangement.

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \cdots$$

$$= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \cdots$$

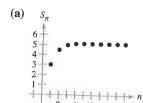$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \cdots$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots\right) = \frac{1}{2}(\ln 2)$$
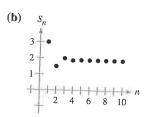
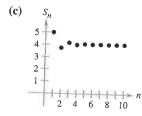By rearranging the terms, you obtain a sum that is half the original sum.
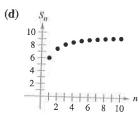
## Exercises for Section 9.5
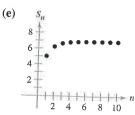
In Exercises 1–6, match the series with the graph of its sequence of partial sums. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

(a)



(b)



(c)



(d)



(e)



(f)



1. $\displaystyle\sum_{n=1}^{\infty} \frac{6}{n^2}$
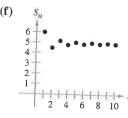
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}6}{n^2}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n!}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}3}{n!}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{n2^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}10}{n2^n}$

*Numerical and Graphical Analysis*  In Exercises 7–10, explore the Alternating Series Remainder.

(a) Use a graphing utility to find the indicated partial sum $S_n$ and complete the table.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|---|---|---|---|----|
| $S_n$ | | | | | | | | | | |

(b) Use a graphing utility to graph the first 10 terms of the sequence of partial sums and a horizontal line representing the sum.

(c) What pattern exists between the plot of the successive points in part (b) relative to the horizontal line representing the sum of the series? Do the distances between the successive points and the horizontal line increase or decrease?

(d) Discuss the relationship between the answers in part (c) and the Alternating Series Remainder as given in Theorem 9.15.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} = \frac{1}{e}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} = \sin 1$

In Exercises 11–32, determine the convergence or divergence of the series.

**11.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

**12.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n - 1}$

**13.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n - 1}$

**14.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n + 1)}$

**15.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2 + 1}$

**16.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2 + 1}$

**17.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

**18.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{n^2 + 5}$

**19.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (n + 1)}{\ln(n + 1)}$

**20.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \ln(n + 1)}{n + 1}$

**21.** $\displaystyle\sum_{n=1}^{\infty} \sin \frac{(2n - 1)\pi}{2}$

**22.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{(2n - 1)\pi}{2}$

**23.** $\displaystyle\sum_{n=1}^{\infty} \cos n\pi$

**24.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \cos n\pi$

**25.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

**26.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!}$

**27.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{n + 2}$

**28.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sqrt{n}}{\sqrt[3]{n}}$

**29.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n - 1)}$

**30.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{1 \cdot 4 \cdot 7 \cdots (3n - 2)}$

**31.** $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n - e^{-n}} = \sum_{n=1}^{\infty} (-1)^{n+1} \operatorname{csch} n$

**32.** $\displaystyle\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{e^n + e^{-n}} = \sum_{n=1}^{\infty} (-1)^{n+1} \operatorname{sech} n$

In Exercises 33–36, approximate the sum of the series by using the first six terms. (See Example 4.)

**33.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3}{n^2}$

**34.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4}{\ln(n + 1)}$

**35.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2}{n!}$

**36.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2^n}$

In Exercises 37–42, (a) use Theorem 9.15 to determine the number of terms required to approximate the sum of the convergent series with an error of less than 0.001, and (b) use a graphing utility to approximate the sum of the series with an error of less than 0.001.

**37.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$

**38.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} = \frac{1}{\sqrt{e}}$

**39.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} = \sin 1$

**40.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} = \cos 1$

**41.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$

**42.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n} = \ln \frac{5}{4}$

In Exercises 43–46, use Theorem 9.15 to determine the number of terms required to approximate the sum of the series with an error of less than 0.001.

**43.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$

**44.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

**45.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 1}$

**46.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$

In Exercises 47–62, determine whether the series converges conditionally or absolutely, or diverges.

**47.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n + 1)^2}$

**48.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n + 1}$

**49.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

**50.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

**51.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(n + 1)^2}$

**52.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n + 3)}{n + 10}$

**53.** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

**54.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n e^{-n^2}$

**55.** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^3 - 1}$

**56.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1.5}}$

**57.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!}$

**58.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n + 4}}$

**59.** $\displaystyle\sum_{n=0}^{\infty} \frac{\cos n\pi}{n + 1}$
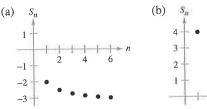
**60.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \arctan n$
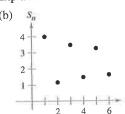
**61.** $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2}$

**62.** $\displaystyle\sum_{n=1}^{\infty} \frac{\sin[(2n - 1)\pi/2]}{n}$

## Writing About Concepts

**63.** Define an alternating series and state the Alternating Series Test.

**64.** Give the remainder after $N$ terms of a convergent alternating series.

**65.** In your own words, state the difference between absolute and conditional convergence of an alternating series.

**66.** The graphs of the sequences of partial sums of two series are shown in the figures. Which graph represents the partial sums of an alternating series? Explain.

(a) $S_n$

(b) $S_n$

**In Exercises 37–50, use the Root Test to determine the convergence or divergence of the series.**

37. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$

38. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$

39. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{2n+1}{n-1}\right)^n$

40. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+3}{2n-1}\right)^n$

41. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

42. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-3n}{2n+1}\right)^{3n}$

43. $\displaystyle\sum_{n=1}^{\infty} \left(2\sqrt[n]{n}+1\right)^n$

44. $\displaystyle\sum_{n=0}^{\infty} e^{-n}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}$

46. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{500}\right)^n$

47. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right)^n$

48. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^n$

49. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

50. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{(n^n)^2}$

**In Exercises 51–68, determine the convergence or divergence of the series using any appropriate test from this chapter. Identify the test used.**

51. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}5}{n}$

52. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n}$

53. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$

54. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$

55. $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{n+1}$

56. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2+1}$

57. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-2}}{2^n}$

58. $\displaystyle\sum_{n=1}^{\infty} \frac{10}{3\sqrt{n^3}}$

59. $\displaystyle\sum_{n=1}^{\infty} \frac{10n+3}{n2^n}$

60. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{4n^2-1}$

61. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$

62. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$

63. $\displaystyle\sum_{n=1}^{\infty} \frac{n7^n}{n!}$

64. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

65. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^{n-1}}{n!}$

66. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n2^n}$

67. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{3\cdot5\cdot7\cdots(2n+1)}$

68. $\displaystyle\sum_{n=1}^{\infty} \frac{3\cdot5\cdot7\cdots(2n+1)}{18^n(2n-1)n!}$

**In Exercises 69–72, identify the two series that are the same.**

69. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n5^n}{n!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n5^n}{(n+1)!}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)5^{n+1}}{(n+1)!}$

70. (a) $\displaystyle\sum_{n=4}^{\infty} n\left(\frac{3}{4}\right)^n$

(b) $\displaystyle\sum_{n=0}^{\infty} (n+1)\left(\frac{3}{4}\right)^n$

(c) $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^{n-1}$

71. (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n+1)!}$

72. (a) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)2^{n-1}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n2^n}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)2^n}$

**In Exercises 73 and 74, write an equivalent series with the index of summation beginning at $n = 0$.**

73. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}$

74. $\displaystyle\sum_{n=2}^{\infty} \frac{2^n}{(n-2)!}$

**In Exercises 75 and 76, (a) determine the number of terms required to approximate the sum of the series with an error less than 0.0001, and (b) use a graphing utility to approximate the sum of the series with an error less than 0.0001.**

75. $\displaystyle\sum_{k=1}^{\infty} \frac{(-3)^k}{2^k k!}$

76. $\displaystyle\sum_{k=0}^{\infty} \frac{(-3)^k}{1\cdot3\cdot5\cdots(2k+1)}$

**In Exercises 77–82, the terms of a series $\displaystyle\sum_{n=1}^{\infty} a_n$ are defined recursively. Determine the convergence or divergence of the series. Explain your reasoning.**

77. $a_1 = \dfrac{1}{2}, \; a_{n+1} = \dfrac{4n-1}{3n+2} a_n$

78. $a_1 = 2, \; a_{n+1} = \dfrac{2n+1}{5n-4} a_n$

79. $a_1 = 1, \; a_{n+1} = \dfrac{\sin n + 1}{\sqrt{n}} a_n$

80. $a_1 = \dfrac{1}{5}, \; a_{n+1} = \dfrac{\cos n + 1}{n} a_n$

81. $a_1 = \dfrac{1}{3}, \; a_{n+1} = \left(1+\dfrac{1}{n}\right) a_n$

82. $a_1 = \dfrac{1}{4}, \; a_{n+1} = \sqrt[n]{a_n}$

**In Exercises 83–86, use the Ratio Test or the Root Test to determine the convergence or divergence of the series.**

83. $1 + \dfrac{1\cdot2}{1\cdot3} + \dfrac{1\cdot2\cdot3}{1\cdot3\cdot5} + \dfrac{1\cdot2\cdot3\cdot4}{1\cdot3\cdot5\cdot7} + \cdots$

84. $1 + \dfrac{2}{3} + \dfrac{3}{3^2} + \dfrac{4}{3^3} + \dfrac{5}{3^4} + \dfrac{6}{3^5} + \cdots$

85. $\dfrac{1}{(\ln 3)^3} + \dfrac{1}{(\ln 4)^4} + \dfrac{1}{(\ln 5)^5} + \dfrac{1}{(\ln 6)^6} + \cdots$

86. $1 + \dfrac{1\cdot3}{1\cdot2\cdot3} + \dfrac{1\cdot3\cdot5}{1\cdot2\cdot3\cdot4\cdot5}$
$+ \dfrac{1\cdot3\cdot5\cdot7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} + \cdots$

**In Exercises 87–92, find the values of $x$ for which the series converges.**

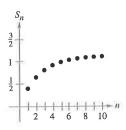**87.** $\displaystyle\sum_{n=0}^{\infty} 2\left(\frac{x}{3}\right)^n$

**88.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x+1}{4}\right)^n$

**89.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x+1)^n}{n}$

**90.** $\displaystyle\sum_{n=0}^{\infty} 2(x-1)^n$

**91.** $\displaystyle\sum_{n=0}^{\infty} n!\left(\frac{x}{2}\right)^n$

**92.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$

## Writing About Concepts

**93.** State the Ratio Test.

**94.** State the Root Test.

**95.** You are told that the terms of a positive series appear to approach zero rapidly as $n$ approaches infinity. In fact, $a_7 \le 0.0001$. Given no other information, does this imply that the series converges? Support your conclusion with examples.

**96.** The graph shows the first 10 terms of the sequence of partial sums of the convergent series

$$\sum_{n=1}^{\infty} \left(\frac{2n}{3n+2}\right)^n.$$

Find a series such that the terms of its sequence of partial sums are less than the corresponding terms of the sequence in the figure, but such that the series diverges. Explain your reasoning.



**97.** Using the Ratio Test, it is determined that an alternating series converges. Does the series converge conditionally or absolutely? Explain.

**98.** Prove Property 2 of Theorem 9.17.

**99.** Prove Theorem 9.18. (*Hint for Property 1:* If the limit equals $r < 1$, choose a real number $R$ such that $r < R < 1$. By the definitions of the limit, there exists some $N > 0$ such that $\sqrt[n]{|a_n|} < R$ for $n > N$.)

**100.** Show that the Root Test is inconclusive for the $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

**101.** Show that the Ratio Test and the Root Test are both inconclusive for the logarithmic $p$-series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}.$$

**102.** Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(xn)!}$$

when (a) $x = 1$, (b) $x = 2$, (c) $x = 3$, and (d) $x$ is a positive integer.

**103.** Show that if $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then

$$\left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|.$$

**104.** *Writing* Read the article "A Differentiation Test for Absolute Convergence" by Yaser S. Abu-Mostafa in *Mathematics Magazine*. Then write a paragraph that describes the test. Include examples of series that converge and examples of series that diverge.

## Putnam Exam Challenge

**105.** Is the following series convergent or divergent?

$$1 + \frac{1}{2}\cdot\frac{19}{7} + \frac{2!}{3^2}\left(\frac{19}{7}\right)^2 + \frac{3!}{4^3}\left(\frac{19}{7}\right)^3 + \frac{4!}{5^4}\left(\frac{19}{7}\right)^4 + \cdots$$

**106.** Show that if the series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

converges, then the series

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} + \cdots$$
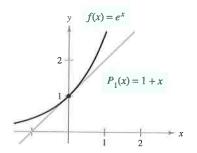
converges also.

# Taylor Polynomials and Approximations

- Find polynomial approximations of elementary functions and compare them with the elementary functions.
- Find Taylor and Maclaurin polynomial approximations of elementary functions.
- Use the remainder of a Taylor polynomial.

## Polynomial Approximations of Elementary Functions

The goal of this section is to show how polynomial functions can be used as approximations for other elementary functions. To find a polynomial function $P$ that approximates another function $f$, begin by choosing a number $c$ in the domain of $f$ at which $f$ and $P$ have the same value. That is,

$$P(c) = f(c). \qquad \text{Graphs of } f \text{ and } P \text{ pass through } (c, f(c)).$$

The approximating polynomial is said to be **expanded about** $c$ or **centered at** $c$. Geometrically, the requirement that $P(c) = f(c)$ means that the graph of $P$ passes through the point $(c, f(c))$. Of course, there are many polynomials whose graphs pass through the point $(c, f(c))$. Your task is to find a polynomial whose graph resembles the graph of $f$ near this point. One way to do this is to impose the additional requirement that the slope of the polynomial function be the same as the slope of the graph of $f$ at the point $(c, f(c))$.

$$P'(c) = f'(c) \qquad \text{Graphs of } f \text{ and } P \text{ have the same slope at } (c, f(c)).$$

With these two requirements, you can obtain a simple linear approximation of $f$, as shown in Figure 9.10.

Near $(c, f(c))$, the graph of $P$ can be used to approximate the graph of $f$.
**Figure 9.10**

### EXAMPLE 1   First-Degree Polynomial Approximation of $f(x) = e^x$

For the function $f(x) = e^x$, find a first-degree polynomial function

$$P_1(x) = a_0 + a_1 x$$

whose value and slope agree with the value and slope of $f$ at $x = 0$.

**Solution**   Because $f(x) = e^x$ and $f'(x) = e^x$, the value and the slope of $f$, at $x = 0$, are given by

$$f(0) = e^0 = 1$$

and

$$f'(0) = e^0 = 1.$$

Because $P_1(x) = a_0 + a_1 x$, you can use the condition that $P_1(0) = f(0)$ to conclude that $a_0 = 1$. Moreover, because $P_1'(x) = a_1$, you can use the condition that $P_1'(0) = f'(0)$ to conclude that $a_1 = 1$. Therefore,

$$P_1(x) = 1 + x.$$

Figure 9.11 shows the graphs of $P_1(x) = 1 + x$ and $f(x) = e^x$. ▬▬▬

$P_1$ is the first-degree polynomial approximation of $f(x) = e^x$.
**Figure 9.11**

NOTE   Example 1 isn't the first time you have used a linear function to approximate another function. The same procedure was used as the basis for Newton's Method.

In Figure 9.12 you can see that, at points near $(0, 1)$, the graph of

$$P_1(x) = 1 + x \qquad \text{\color{red}1st-degree approximation}$$

is reasonably close to the graph of $f(x) = e^x$. However, as you move away from $(0, 1)$, the graphs move farther from each other and the accuracy of the approximation decreases. To improve the approximation, you can impose yet another requirement—that the values of the second derivatives of $P$ and $f$ agree when $x = 0$. The polynomial, $P_2$, of least degree that satisfies all three requirements $P_2(0) = f(0)$, $P_2{}'(0) = f'(0)$, and $P_2{}''(0) = f''(0)$ can be shown to be

$$P_2(x) = 1 + x + \frac{1}{2}x^2. \qquad \text{\color{red}2nd-degree approximation}$$

Moreover, in Figure 9.12, you can see that $P_2$ is a better approximation of $f$ than $P_1$. If you continue this pattern, requiring that the values of $P_n(x)$ and its first $n$ derivatives match those of $f(x) = e^x$ at $x = 0$, you obtain the following.

$$P_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n \qquad \text{\color{red}$n$th-degree approximation}$$

$$\approx e^x$$



$P_2$ is the second-degree polynomial approximation of $f(x) = e^x$.
**Figure 9.12**

## EXAMPLE 2    Third-Degree Polynomial Approximation of $f(x) = e^x$

Construct a table comparing the values of the polynomial
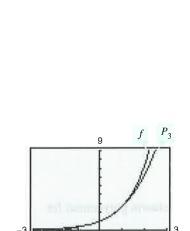
$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 \qquad \text{\color{red}3rd-degree approximation}$$

with $f(x) = e^x$ for several values of $x$ near 0.

**Solution**    Using a calculator or a computer, you can obtain the results shown in the table. Note that for $x = 0$, the two functions have the same value, but that as $x$ moves farther away from 0, the accuracy of the approximating polynomial $P_3(x)$ decreases.

| $x$ | $-1.0$ | $-0.2$ | $-0.1$ | 0 | 0.1 | 0.2 | 1.0 |
|---|---|---|---|---|---|---|---|
| $e^x$ | 0.3679 | 0.81873 | 0.904837 | 1 | 1.105171 | 1.22140 | 2.7183 |
| $P_3(x)$ | 0.3333 | 0.81867 | 0.904833 | 1 | 1.105167 | 1.22133 | 2.6667 |



$P_3$ is the third-degree polynomial approximation of $f(x) = e^x$.
**Figure 9.13**

**TECHNOLOGY**    A graphing utility can be used to compare the graph of the approximating polynomial with the graph of the function $f$. For instance, in Figure 9.13, the graph of

$$P_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \qquad \text{\color{red}3rd-degree approximation}$$

is compared with the graph of $f(x) = e^x$. If you have access to a graphing utility, try comparing the graphs of

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 \qquad \text{\color{red}4th-degree approximation}$$

$$P_5(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 \qquad \text{\color{red}5th-degree approximation}$$

$$P_6(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 \qquad \text{\color{red}6th-degree approximation}$$

with the graph of $f$. What do you notice?

**BROOK TAYLOR (1685–1731)**

Although Taylor was not the first to seek polynomial approximations of transcendental functions, his account published in 1715 was one of the first comprehensive works on the subject.

## Taylor and Maclaurin Polynomials

The polynomial approximation of $f(x) = e^x$ given in Example 2 is expanded about $c = 0$. For expansions about an arbitrary value of $c$, it is convenient to write the polynomial in the form

$$P_n(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \cdots + a_n(x - c)^n.$$

In this form, repeated differentiation produces

$$P_n'(x) = a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + \cdots + na_n(x - c)^{n-1}$$
$$P_n''(x) = 2a_2 + 2(3a_3)(x - c) + \cdots + n(n - 1)a_n(x - c)^{n-2}$$
$$P_n'''(x) = 2(3a_3) + \cdots + n(n - 1)(n - 2)a_n(x - c)^{n-3}$$
$$\vdots$$
$$P_n^{(n)}(x) = n(n - 1)(n - 2) \cdots (2)(1)a_n.$$

Letting $x = c$, you then obtain

$$P_n(c) = a_0, \qquad P_n'(c) = a_1, \qquad P_n''(c) = 2a_2, \quad \ldots, \qquad P_n^{(n)}(c) = n!a_n$$

and because the value of $f$ and its first $n$ derivatives must agree with the value of $P_n$ and its first $n$ derivatives at $x = c$, it follows that

$$f(c) = a_0, \qquad f'(c) = a_1, \qquad \frac{f''(c)}{2!} = a_2, \quad \ldots, \qquad \frac{f^{(n)}(c)}{n!} = a_n.$$

With these coefficients, you can obtain the following definition of **Taylor polynomials,** named after the English mathematician Brook Taylor, and **Maclaurin polynomials,** named after the English mathematician Colin Maclaurin (1698–1746).

**NOTE** Maclaurin polynomials are special types of Taylor polynomials for which $c = 0$.

---

**Definitions of $n$th Taylor Polynomial and $n$th Maclaurin Polynomial**

If $f$ has $n$ derivatives at $c$, then the polynomial

$$P_n(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n$$

is called the **$n$th Taylor polynomial for $f$ at $c$.** If $c = 0$, then

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

is also called the **$n$th Maclaurin polynomial for $f$.**

---

### EXAMPLE 3  A Maclaurin Polynomial for $f(x) = e^x$

Find the $n$th Maclaurin polynomial for $f(x) = e^x$.

**Solution** From the discussion on page 649, the $n$th Maclaurin polynomial for

$$f(x) = e^x$$

is given by

$$P_n(x) = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n.$$

**FOR FURTHER INFORMATION** To see how to use series to obtain other approximations to $e$, see the article "Novel Series-based Approximations to $e$" by John Knox and Harlan J. Brothers in *The College Mathematics Journal.* To view this article, go to the website *www.matharticles.com.*

*EXAMPLE 4*   **Finding Taylor Polynomials for ln x**

Find the Taylor polynomials $P_0$, $P_1$, $P_2$, $P_3$, and $P_4$ for $f(x) = \ln x$ centered at $c = 1$.

**Solution**   Expanding about $c = 1$ yields the following.

$$f(x) = \ln x \qquad\qquad f(1) = \ln 1 = 0$$

$$f'(x) = \frac{1}{x} \qquad\qquad f'(1) = \frac{1}{1} = 1$$

$$f''(x) = -\frac{1}{x^2} \qquad\qquad f''(1) = -\frac{1}{1^2} = -1$$

$$f'''(x) = \frac{2!}{x^3} \qquad\qquad f'''(1) = \frac{2!}{1^3} = 2$$

$$f^{(4)}(x) = -\frac{3!}{x^4} \qquad\qquad f^{(4)}(1) = -\frac{3!}{1^4} = -6$$

Therefore, the Taylor polynomials are as follows.

$$P_0(x) = f(1) = 0$$
$$P_1(x) = f(1) + f'(1)(x - 1) = (x - 1)$$
$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2$$

$$= (x - 1) - \frac{1}{2}(x - 1)^2$$

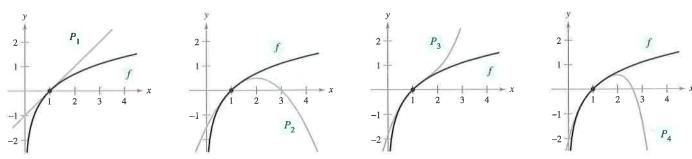$$P_3(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3$$

$$= (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3$$

$$P_4(x) = f(1) + f'(1)(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3$$

$$+ \frac{f^{(4)}(1)}{4!}(x - 1)^4$$

$$= (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4$$

Figure 9.14 compares the graphs of $P_1$, $P_2$, $P_3$, and $P_4$ with the graph of $f(x) = \ln x$. Note that near $x = 1$ the graphs are nearly indistinguishable. For instance, $P_4(0.9) \approx -0.105358$ and $\ln(0.9) \approx -0.105361$.
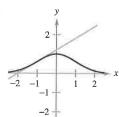


As $n$ increases, the graph of $P_n$ becomes a better and better approximation of the graph of $f(x) = \ln x$ near $x = 1$.
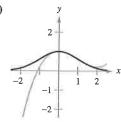**Figure 9.14**
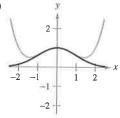
## Exercises for Section 9.7

In Exercises 1–4, match the Taylor polynomial approximation of the function $f(x) = e^{-x^2/2}$ with the correct graph. [The graphs are labeled (a), (b), (c), and (d).]
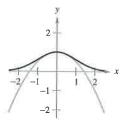
(a)



(b)



(c)



(d)



**1.** $g(x) = -\frac{1}{2}x^2 + 1$

**2.** $g(x) = \frac{1}{8}x^4 - \frac{1}{2}x^2 + 1$

**3.** $g(x) = e^{-1/2}[(x + 1) + 1]$

**4.** $g(x) = e^{-1/2}\left[\frac{1}{3}(x - 1)^3 - (x - 1) + 1\right]$

In Exercises 5–8, find a first-degree polynomial function $P_1$ whose value and slope agree with the value and slope of $f$ at $x = c$. Use a graphing utility to graph $f$ and $P_1$. What is $P_1$ called?

**5.** $f(x) = \dfrac{4}{\sqrt{x}}, \quad c = 1$

**6.** $f(x) = \dfrac{4}{\sqrt[3]{x}}, \quad c = 8$

**7.** $f(x) = \sec x, \quad c = \dfrac{\pi}{4}$

**8.** $f(x) = \tan x, \quad c = \dfrac{\pi}{4}$

*Graphical and Numerical Analysis* In Exercises 9 and 10, use a graphing utility to graph $f$ and its second-degree polynomial approximation $P_2$ at $x = c$. Complete the table comparing the values of $f$ and $P_2$.

**9.** $f(x) = \dfrac{4}{\sqrt{x}}, \quad c = 1$

$P_2(x) = 4 - 2(x - 1) + \frac{3}{2}(x - 1)^2$

| $x$ | 0 | 0.8 | 0.9 | 1 | 1.1 | 1.2 | 2 |
|------|---|-----|-----|---|-----|-----|---|
| $f(x)$ | | | | | | | |
| $P_2(x)$ | | | | | | | |

**10.** $f(x) = \sec x, \quad c = \dfrac{\pi}{4}$

$$P_2(x) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) + \frac{3}{2}\sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$

| $x$ | $-2.15$ | 0.585 | 0.685 | $\dfrac{\pi}{4}$ | 0.885 | 0.985 | 1.785 |
|------|---------|-------|-------|------|-------|-------|-------|
| $f(x)$ | | | | | | | |
| $P_2(x)$ | | | | | | | |

**11.** *Conjecture* Consider the function $f(x) = \cos x$ and its Maclaurin polynomials $P_2$, $P_4$, and $P_6$ (see Example 5).

(a) Use a graphing utility to graph $f$ and the indicated polynomial approximations.

(b) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2, 4,$ and 6.

(c) Use the results in part (b) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

**12.** *Conjecture* Consider the function $f(x) = x^2 e^x$.

(a) Find the Maclaurin polynomials $P_2$, $P_3$, and $P_4$ for $f$.

(b) Use a graphing utility to graph $f$, $P_2$, $P_3$, and $P_4$.

(c) Evaluate and compare the values of $f^{(n)}(0)$ and $P_n^{(n)}(0)$ for $n = 2, 3,$ and 4.

(d) Use the results in part (c) to make a conjecture about $f^{(n)}(0)$ and $P_n^{(n)}(0)$.

In Exercises 13–24, find the Maclaurin polynomial of degree $n$ for the function.

**13.** $f(x) = e^{-x}, \quad n = 3$

**14.** $f(x) = e^{-x}, \quad n = 5$

**15.** $f(x) = e^{2x}, \quad n = 4$

**16.** $f(x) = e^{3x}, \quad n = 4$

**17.** $f(x) = \sin x, \quad n = 5$

**18.** $f(x) = \sin \pi x, \quad n = 3$

**19.** $f(x) = xe^x, \quad n = 4$

**20.** $f(x) = x^2 e^{-x}, \quad n = 4$

**21.** $f(x) = \dfrac{1}{x + 1}, \quad n = 4$

**22.** $f(x) = \dfrac{x}{x + 1}, \quad n = 4$

**23.** $f(x) = \sec x, \quad n = 2$

**24.** $f(x) = \tan x, \quad n = 3$

In Exercises 25–30, find the $n$th Taylor polynomial centered at $c$.

**25.** $f(x) = \dfrac{1}{x}, \quad n = 4, \quad c = 1$

**26.** $f(x) = \dfrac{2}{x^2}, \quad n = 4, \quad c = 2$

**27.** $f(x) = \sqrt{x}, \quad n = 4, \quad c = 1$

**28.** $f(x) = \sqrt[3]{x}, \quad n = 3, \quad c = 8$

**29.** $f(x) = \ln x, \quad n = 4, \quad c = 1$

**30.** $f(x) = x^2 \cos x, \quad n = 2, \quad c = \pi$

In Exercises 31 and 32, use a computer algebra system to find the indicated Taylor polynomials for the function $f$. Graph the function and the Taylor polynomials.

**31.** $f(x) = \tan x$

(a) $n = 3$, $c = 0$

(b) $n = 3$, $c = \pi/4$

**32.** $f(x) = \dfrac{1}{x^2 + 1}$

(a) $n = 4$, $c = 0$

(b) $n = 4$, $c = 1$

**33.** *Numerical and Graphical Approximations*

(a) Use the Maclaurin polynomials $P_1(x)$, $P_3(x)$, and $P_5(x)$ for $f(x) = \sin x$ to complete the table.

| $x$ | 0 | 0.25 | 0.50 | 0.75 | 1.00 |
|---|---|---|---|---|---|
| $\sin x$ | 0 | 0.2474 | 0.4794 | 0.6816 | 0.8415 |
| $P_1(x)$ | | | | | |
| $P_3(x)$ | | | | | |
| $P_5(x)$ | | | | | |

(b) Use a graphing utility to graph $f(x) = \sin x$ and the Maclaurin polynomials in part (a).

(c) Describe the change in accuracy of a polynomial approximation as the distance from the point where the polynomial is centered increases.

**34.** *Numerical and Graphical Approximations*

(a) Use the Taylor polynomials $P_1(x)$, $P_2(x)$, and $P_4(x)$ for $f(x) = \ln x$ centered at $c = 1$ to complete the table.

| $x$ | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 |
|---|---|---|---|---|---|
| $\ln x$ | 0 | 0.2231 | 0.4055 | 0.5596 | 0.6931 |
| $P_1(x)$ | | | | | |
| $P_2(x)$ | | | | | |
| $P_4(x)$ | | | | | |

(b) Use a graphing utility to graph $f(x) = \ln x$ and the Taylor polynomials in part (a).

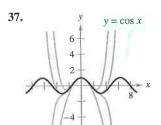(c) Describe the change in accuracy of polynomial approximations as the degree increases.

*Numerical and Graphical Approximations*   In Exercises 35 and 36, (a) find the Maclaurin polynomial $P_3(x)$ for $f(x)$, (b) complete the table for $f(x)$ and $P_3(x)$, and (c) sketch the graphs of $f(x)$ and $P_3(x)$ on the same set of coordinate axes.
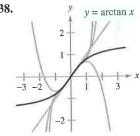
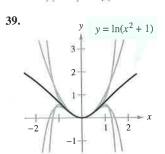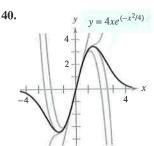| $x$ | −0.75 | −0.50 | −0.25 | 0 | 0.25 | 0.50 | 0.75 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | | |
| $P_3(x)$ | | | | | | | |

**35.** $f(x) = \arcsin x$

**36.** $f(x) = \arctan x$

In Exercises 37–40, the graph of $y = f(x)$ is shown with four of its Maclaurin polynomials. Identify the Maclaurin polynomials and use a graphing utility to confirm your results.

**37.**



$y = \cos x$

**38.**



$y = \arctan x$

**39.**



$y = \ln(x^2 + 1)$

**40.**



$y = 4xe^{(-x^2/4)}$

In Exercises 41–44, approximate the function at the given value of $x$, using the polynomial found in the indicated exercise.

**41.** $f(x) = e^{-x}$,   $f\left(\tfrac{1}{2}\right)$, Exercise 13

**42.** $f(x) = x^2 e^{-x}$,   $f\left(\tfrac{1}{5}\right)$, Exercise 20

**43.** $f(x) = \ln x$,   $f(1.2)$, Exercise 29

**44.** $f(x) = x^2 \cos x$,   $f\left(\dfrac{7\pi}{8}\right)$, Exercise 30

In Exercises 45–48, use Taylor's Theorem to obtain an upper bound for the error of the approximation. Then calculate the exact value of the error.

**45.** $\cos(0.3) \approx 1 - \dfrac{(0.3)^2}{2!} + \dfrac{(0.3)^4}{4!}$

**46.** $e \approx 1 + 1 + \dfrac{1^2}{2!} + \dfrac{1^3}{3!} + \dfrac{1^4}{4!} + \dfrac{1^5}{5!}$

**47.** $\arcsin(0.4) \approx 0.4 + \dfrac{(0.4)^3}{2 \cdot 3}$   **48.** $\arctan(0.4) \approx 0.4 - \dfrac{(0.4)^3}{3}$

In Exercises 49–52, determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of $x$ to be less than 0.001.

**49.** $\sin(0.3)$   **50.** $\cos(0.1)$

**51.** $e^{0.6}$   **52.** $e^{0.3}$

In Exercises 53–56, determine the degree of the Maclaurin polynomial required for the error in the approximation of the function at the indicated value of $x$ to be less than 0.0001. Use a computer algebra system to obtain and evaluate the required derivatives.

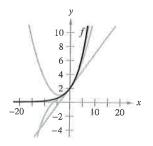**53.** $f(x) = \ln(x + 1)$, approximate $f(0.5)$.

**54.** $f(x) = \cos(\pi x^2)$, approximate $f(0.6)$.

**55.** $f(x) = e^{-\pi x}$, approximate $f(1.3)$.

**56.** $f(x) = e^{-x}$, approximate $f(1)$.

In Exercises 57–60, determine the values of $x$ for which the function can be replaced by the Taylor polynomial if the error cannot exceed 0.001.

**57.** $f(x) = e^x \approx 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$,    $x < 0$

**58.** $f(x) = \sin x \approx x - \dfrac{x^3}{3!}$

**59.** $f(x) = \cos x \approx 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}$

**60.** $f(x) = e^{-2x} \approx 1 - 2x + 2x^2 - \dfrac{4}{3}x^3$

## Writing About Concepts

**61.** An elementary function is approximated by a polynomial. In your own words, describe what is meant by saying that the polynomial is *expanded about c* or *centered at c*.

**62.** When an elementary function $f$ is approximated by a second-degree polynomial $P_2$ centered at $c$, what is known about $f$ and $P_2$ at $c$? Explain your reasoning.

**63.** State the definition of an $n$th-degree Taylor polynomial of $f$ centered at $c$.

**64.** Describe the accuracy of the $n$th-degree Taylor polynomial of $f$ centered at $c$ as the distance between $c$ and $x$ increases.

**65.** In general, how does the accuracy of a Taylor polynomial change as the degree of the polynomial is increased? Explain your reasoning.

**66.** The graphs show first-, second-, and third-degree polynomial approximations $P_1$, $P_2$, and $P_3$ of a function $f$. Label the graphs of $P_1$, $P_2$, and $P_3$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



**67.** *Comparing Maclaurin Polynomials*

(a) Compare the Maclaurin polynomials of degree 4 and degree 5, respectively, for the functions $f(x) = e^x$ and $g(x) = xe^x$. What is the relationship between them?

(b) Use the result in part (a) and the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ to find a Maclaurin polynomial of degree 6 for the function $g(x) = x \sin x$.

(c) Use the result in part (a) and the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ to find a Maclaurin polynomial of degree 4 for the function $g(x) = (\sin x)/x$.
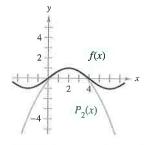
**68.** *Differentiating Maclaurin Polynomials*

(a) Differentiate the Maclaurin polynomial of degree 5 for $f(x) = \sin x$ and compare the result with the Maclaurin polynomial of degree 4 for $g(x) = \cos x$.

(b) Differentiate the Maclaurin polynomial of degree 6 for $f(x) = \cos x$ and compare the result with the Maclaurin polynomial of degree 5 for $g(x) = \sin x$.

(c) Differentiate the Maclaurin polynomial of degree 4 for $f(x) = e^x$. Describe the relationship between the two series.

**69.** *Graphical Reasoning*   The figure shows the graph of the function

$$f(x) = \sin\left(\frac{\pi x}{4}\right)$$

and the second-degree Taylor polynomial

$$P_2(x) = 1 - \frac{\pi^2}{32}(x - 2)^2$$

centered at $x = 2$.



(a) Use the symmetry of the graph of $f$ to write the second-degree Taylor polynomial for $f$ centered at $x = -2$.

(b) Use a horizontal translation of the result in part (a) to find the second-degree Taylor polynomial for $f$ centered at $x = 6$.

(c) Is it possible to use a horizontal translation of the result in part (a) to write a second-degree Taylor polynomial for $f$ centered at $x = 4$? Explain.

**70.** Prove that if $f$ is an odd function, then its $n$th Maclaurin polynomial contains only terms with odd powers of $x$.

**71.** Prove that if $f$ is an even function, then its $n$th Maclaurin polynomial contains only terms with even powers of $x$.

**72.** Let $P_n(x)$ be the $n$th Taylor polynomial for $f$ at $c$. Prove that $P_n(c) = f(c)$ and $P^{(k)}(c) = f^{(k)}(c)$ for $1 \le k \le n$. (See Exercises 9 and 10.)

**73.** *Writing*   The proof in Exercise 72 guarantees that the Taylor polynomial and its derivatives agree with the function and its derivatives at $x = c$. Use the graphs and tables in Exercises 33–36 to discuss what happens to the accuracy of the Taylor polynomial as you move away from $x = c$.

**Section 9.8**

## Power Series

- Understand the definition of a power series.
- Find the radius and interval of convergence of a power series.
- Determine the endpoint convergence of a power series.
- Differentiate and integrate a power series.

### Power Series

In Section 9.7, you were introduced to the concept of approximating functions by Taylor polynomials. For instance, the function $f(x) = e^x$ can be *approximated* by its Maclaurin polynomials as follows.

$$e^x \approx 1 + x \qquad\qquad \text{1st-degree polynomial}$$

$$e^x \approx 1 + x + \frac{x^2}{2!} \qquad\qquad \text{2nd-degree polynomial}$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \qquad\qquad \text{3rd-degree polynomial}$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad\qquad \text{4th-degree polynomial}$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \qquad\qquad \text{5th-degree polynomial}$$

In that section, you saw that the higher the degree of the approximating polynomial, the better the approximation becomes.

In this and the next two sections, you will see that several important types of functions, including

$$f(x) = e^x$$

can be represented *exactly* by an infinite series called a **power series.** For example, the power series representation for $e^x$ is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots.$$

For each real number $x$, it can be shown that the infinite series on the right converges to the number $e^x$. Before doing this, however, some preliminary results dealing with power series will be discussed—beginning with the following definition.

---

**EXPLORATION**

*Graphical Reasoning* Use a graphing utility to approximate the graph of each power series near $x = 0$. (Use the first several terms of each series.) Each series represents a well-known function. What is the function?

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

c. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

d. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

e. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

---

### Definition of Power Series

If $x$ is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

is called a **power series.** More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + \cdots + a_n(x - c)^n + \cdots$$

is called a **power series centered at $c$,** where $c$ is a constant.

---

**NOTE**  To simplify the notation for power series, we agree that $(x - c)^0 = 1$, even if $x = c$.

## EXAMPLE I Power Series

**a.** The following power series is centered at 0.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

**b.** The following power series is centered at $-1$.

$$\sum_{n=0}^{\infty} (-1)^n (x + 1)^n = 1 - (x + 1) + (x + 1)^2 - (x + 1)^3 + \cdots$$

**c.** The following power series is centered at 1.

$$\sum_{n=1}^{\infty} \frac{1}{n} (x - 1)^n = (x - 1) + \frac{1}{2} (x - 1)^2 + \frac{1}{3} (x - 1)^3 + \cdots$$

## Radius and Interval of Convergence

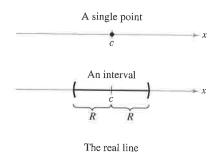A power series in $x$ can be viewed as a function of $x$

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$

where the *domain of $f$* is the set of all $x$ for which the power series converges. Determination of the domain of a power series is the primary concern in this section. Of course, every power series converges at its center $c$ because

$$f(c) = \sum_{n=0}^{\infty} a_n(c - c)^n$$
$$= a_0(1) + 0 + 0 + \cdots + 0 + \cdots$$
$$= a_0.$$

So, $c$ always lies in the domain of $f$. The following important theorem states that the domain of a power series can take three basic forms: a single point, an interval centered at $c$, or the entire real line, as shown in Figure 9.17. A proof is given in Appendix A.

A single point

An interval

The real line

The domain of a power series has only three basic forms: a single point, an interval centered at $c$, or the entire real line.
**Figure 9.17**

---

### THEOREM 9.20 Convergence of a Power Series

For a power series centered at $c$, precisely one of the following is true.

1. The series converges only at $c$.
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$, and diverges for $|x - c| > R$.
3. The series converges absolutely for all $x$.

The number $R$ is the **radius of convergence** of the power series. If the series converges only at $c$, the radius of convergence is $R = 0$, and if the series converges for all $x$, the radius of convergence is $R = \infty$. The set of all values of $x$ for which the power series converges is the **interval of convergence** of the power series.

## EXAMPLE 2    Finding the Radius of Convergence

Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} n!x^n$.

**Solution**   For $x = 0$, you obtain

$$f(0) = \sum_{n=0}^{\infty} n!0^n = 1 + 0 + 0 + \cdots = 1.$$

For any fixed value of $x$ such that $|x| > 0$, let $u_n = n!x^n$. Then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right|$$

$$= |x| \lim_{n\to\infty} (n+1)$$

$$= \infty.$$

Therefore, by the Ratio Test, the series diverges for $|x| > 0$ and converges only at its
center, 0. So, the radius of convergence is $R = 0$.

## EXAMPLE 3    Finding the Radius of Convergence

Find the radius of convergence of

$$\sum_{n=0}^{\infty} 3(x - 2)^n.$$

**Solution**   For $x \neq 2$, let $u_n = 3(x - 2)^n$. Then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{3(x-2)^{n+1}}{3(x-2)^n}\right|$$

$$= \lim_{n\to\infty} |x - 2|$$

$$= |x - 2|.$$

By the Ratio Test, the series converges if $|x - 2| < 1$ and diverges if $|x - 2| > 1$.
Therefore, the radius of convergence of the series is $R = 1$.

## EXAMPLE 4    Finding the Radius of Convergence

Find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

**Solution**   Let $u_n = (-1)^n x^{2n+1}/(2n+1)!$. Then

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{\dfrac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}}{\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}}\right|$$

$$= \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)}.$$

For any *fixed* value of $x$, this limit is 0. So, by the Ratio Test, the series converges for
all $x$. Therefore, the radius of convergence is $R = \infty$.

## Endpoint Convergence

Note that for a power series whose radius of convergence is a finite number $R$, Theorem 9.20 says nothing about the convergence at the *endpoints* of the interval of convergence. Each endpoint must be tested separately for convergence or divergence. As a result, the interval of convergence of a power series can take any one of the six forms shown in Figure 9.18.
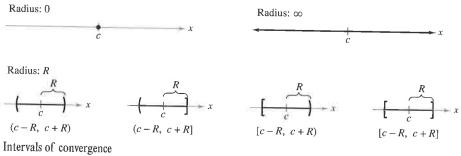
Radius: 0

Radius: $\infty$

Radius: $R$

$(c - R, \ c + R)$   $(c - R, \ c + R]$   $[c - R, \ c + R)$   $[c - R, \ c + R]$

Intervals of convergence
**Figure 9.18**

## EXAMPLE 5  Finding the Interval of Convergence

Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$.

**Solution**   Letting $u_n = x^n/n$ produces

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{x^{n+1}}{(n+1)}}{\dfrac{x^n}{n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{nx}{n+1} \right|$$

$$= |x|.$$

So, by the Ratio Test, the radius of convergence is $R = 1$. Moreover, because the series is centered at 0, it converges in the interval $(-1, 1)$. This interval, however, is not necessarily the *interval of convergence*. To determine this, you must test for convergence at each endpoint. When $x = 1$, you obtain the *divergent* harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots .$$   Diverges when $x = 1$

When $x = -1$, you obtain the *convergent* alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots .$$   Converges when $x = -1$

So, the interval of convergence for the series is $[-1, 1)$, as shown in Figure 9.19.

Interval: $[-1, 1)$
Radius: $R = 1$

**Figure 9.19**

## EXAMPLE 6    Finding the Interval of Convergence

Find the interval of convergence of $\sum_{n=0}^{\infty} \dfrac{(-1)^n(x+1)^n}{2^n}$.

**Solution**    Letting $u_n = (-1)^n(x+1)^n/2^n$ produces

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}(x+1)^{n+1}}{2^{n+1}}}{\dfrac{(-1)^n(x+1)^n}{2^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{2^n(x+1)}{2^{n+1}} \right|$$

$$= \left| \frac{x+1}{2} \right|.$$

By the Ratio Test, the series converges if $|(x+1)/2| < 1$ or $|x+1| < 2$. So, the radius of convergence is $R = 2$. Because the series is centered at $x = -1$, it will converge in the interval $(-3, 1)$. Furthermore, at the endpoints you have

$$\sum_{n=0}^{\infty} \frac{(-1)^n(-2)^n}{2^n} = \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1 \qquad \textcolor{red}{\text{Diverges when } x = -3}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n(2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n \qquad \textcolor{red}{\text{Diverges when } x = 1}$$

both of which diverge. So, the interval of convergence is $(-3, 1)$, as shown in Figure 9.20.
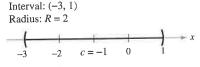
Interval: $(-3, 1)$
Radius: $R = 2$



**Figure 9.20**

## EXAMPLE 7    Finding the Interval of Convergence

Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

**Solution**    Letting $u_n = x^n/n^2$ produces

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)^2}{x^n/n^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2 x}{(n+1)^2} \right| = |x|.$$

So, the radius of convergence is $R = 1$. Because the series is centered at $x = 0$, it converges in the interval $(-1, 1)$. When $x = 1$, you obtain the convergent $p$-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots. \qquad \textcolor{red}{\text{Converges when } x = 1}$$

When $x = -1$, you obtain the convergent alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots. \qquad \textcolor{red}{\text{Converges when } x = -1}$$

Therefore, the interval of convergence for the given series is $[-1, 1]$.

**JAMES GREGORY (1638–1675)**

One of the earliest mathematicians to work with power series was a Scotsman, James Gregory. He developed a power series method for interpolating table values—a method that was later used by Brook Taylor in the development of Taylor polynomials and Taylor series.

## Differentiation and Integration of Power Series

Power series representation of functions has played an important role in the development of calculus. In fact, much of Newton's work with differentiation and integration was done in the context of power series—especially his work with complicated algebraic functions and transcendental functions. Euler, Lagrange, Leibniz, and the Bernoullis all used power series extensively in calculus.

Once you have defined a function with a power series, it is natural to wonder how you can determine the characteristics of the function. Is it continuous? Differentiable? Theorem 9.21, which is stated without proof, answers these questions.

---

**THEOREM 9.21** **Properties of Functions Defined by Power Series**

If the function given by

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$$
$$= a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \cdots$$

has a radius of convergence of $R > 0$, then, on the interval $(c - R, c + R)$, $f$ is differentiable (and therefore continuous). Moreover, the derivative and antiderivative of $f$ are as follows.

**1.** $f'(x) = \sum_{n=1}^{\infty} na_n(x - c)^{n-1}$
$$= a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + \cdots$$

**2.** $\int f(x)\, dx = C + \sum_{n=0}^{\infty} a_n \frac{(x - c)^{n+1}}{n + 1}$
$$= C + a_0(x - c) + a_1 \frac{(x - c)^2}{2} + a_2 \frac{(x - c)^3}{3} + \cdots$$

The *radius of convergence* of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The *interval of convergence*, however, may differ as a result of the behavior at the endpoints.

---

Theorem 9.21 states that, in many ways, a function defined by a power series behaves like a polynomial. It is continuous in its interval of convergence, and both its derivative and its antiderivative can be determined by differentiating and integrating each term of the given power series. For instance, the derivative of the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

is

$$f'(x) = 1 + (2)\frac{x}{2} + (3)\frac{x^2}{3!} + (4)\frac{x^3}{4!} + \cdots$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$
$$= f(x).$$

Notice that $f'(x) = f(x)$. Do you recognize this function?

## EXAMPLE 8    Intervals of Convergence for $f(x)$, $f'(x)$, and $\int f(x)\,dx$

Consider the function given by

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots .$$

Find the intervals of convergence for each of the following.

**a.** $\int f(x)\,dx$     **b.** $f(x)$     **c.** $f'(x)$

**Solution**   By Theorem 9.21, you have

$$f'(x) = \sum_{n=1}^{\infty} x^{n-1}$$
$$= 1 + x + x^2 + x^3 + \cdots$$

and

$$\int f(x)\,dx = C + \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$$
$$= C + \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots .$$

By the Ratio Test, you can show that each series has a radius of convergence of $R = 1$. Considering the interval $(-1, 1)$, you have the following.

**a.** For $\int f(x)\,dx$, the series

$$\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} \qquad \text{Interval of convergence: } [-1, 1]$$

converges for $x = \pm 1$, and its interval of convergence is $[-1, 1]$. See Figure 9.21(a).

**b.** For $f(x)$, the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \qquad \text{Interval of convergence: } [-1, 1)$$

converges for $x = -1$ and diverges for $x = 1$. So, its interval of convergence is $[-1, 1)$. See Figure 9.21(b).

**c.** For $f'(x)$, the series

$$\sum_{n=1}^{\infty} x^{n-1} \qquad \text{Interval of convergence: } (-1, 1)$$

diverges for $x = \pm 1$, and its interval of convergence is $(-1, 1)$. See Figure 9.21(c).

Interval: $[-1, 1]$
Radius: $R = 1$

Interval: $[-1, 1)$
Radius: $R = 1$

Interval: $(-1, 1)$
Radius: $R = 1$

(a)

(b)

(c)

**Figure 9.21**

From Example 8, it appears that of the three series, the one for the derivative, $f'(x)$, is the least likely to converge at the endpoints. In fact, it can be shown that if the series for $f'(x)$ converges at the endpoints $x = c \pm R$, the series for $f(x)$ will also converge there.

In Exercises 1–4, state where the power series is centered.

**1.** $\displaystyle\sum_{n=0}^{\infty} n x^n$

**2.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdots (2n-1)}{2^n n!} x^n$

**3.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^3}$

**4.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi)^{2n}}{(2n)!}$

In Exercises 5–10, find the radius of convergence of the power series.

**5.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}$

**6.** $\displaystyle\sum_{n=0}^{\infty} (2x)^n$

**7.** $\displaystyle\sum_{n=1}^{\infty} \frac{(2x)^n}{n^2}$

**8.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}$

**9.** $\displaystyle\sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!}$

**10.** $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)! x^{2n}}{n!}$

In Exercises 11–34, find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

**11.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

**12.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{5}\right)^n$

**13.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

**14.** $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

**15.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

**16.** $\displaystyle\sum_{n=0}^{\infty} \frac{(3x)^n}{(2n)!}$

**17.** $\displaystyle\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{2}\right)^n$

**18.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$

**19.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{4^n}$

**20.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n n! (x-4)^n}{3^n}$

**21.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n 5^n}$

**22.** $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{(n+1)4^{n+1}}$

**23.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$

**24.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n 2^n}$

**25.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^{n-1}}{3^{n-1}}$

**26.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

**27.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}(-2x)^{n-1}$

**28.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

**29.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

**30.** $\displaystyle\sum_{n=1}^{\infty} \frac{n! x^n}{(2n)!}$

**31.** $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 3 \cdot 4 \cdots (n+1)x^n}{n!}$

**32.** $\displaystyle\sum_{n=1}^{\infty} \left[\frac{2 \cdot 4 \cdot 6 \cdots 2n}{3 \cdot 5 \cdot 7 \cdots (2n+1)}\right] x^{2n+1}$

**33.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3 \cdot 7 \cdot 11 \cdots (4n-1)(x-3)^n}{4^n}$

**34.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x+1)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

In Exercises 35 and 36, find the radius of convergence of the power series, where $c > 0$ and $k$ is a positive integer.

**35.** $\displaystyle\sum_{n=1}^{\infty} \frac{(x-c)^{n-1}}{c^{n-1}}$

**36.** $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^k x^n}{(kn)!}$

In Exercises 37–40, find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)

**37.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{k}\right)^n, \quad k > 0$

**38.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-c)^n}{nc^n}$

**39.** $\displaystyle\sum_{n=1}^{\infty} \frac{k(k+1)(k+2)\cdots(k+n-1)x^n}{n!}, \quad k \geq 1$

**40.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x-c)^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$

In Exercises 41–44, write an equivalent series with the index of summation beginning at $n = 1$.

**41.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

**42.** $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$

**43.** $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

**44.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

In Exercises 45–48, find the intervals of convergence of (a) $f(x)$, (b) $f'(x)$, (c) $f''(x)$, and (d) $\int f(x)\,dx$. Include a check for convergence at the endpoints of the interval.

**45.** $f(x) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$

**46.** $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n 5^n}$
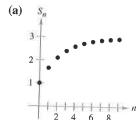
**47.** $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$
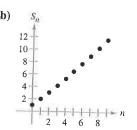
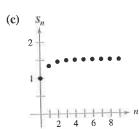**48.** $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-2)^n}{n}$

*Writing* In Exercises 49–52, match the graph of the first 10 terms of the sequence of partial sums of the series
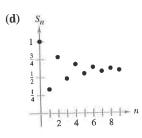
$$g(x) = \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$$

with the indicated value of the function. [The graphs are labeled (a), (b), (c), and (d).] Explain how you made your choice.
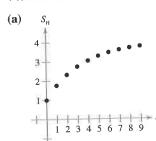
(a)



(b)

**(c)** $S_n$



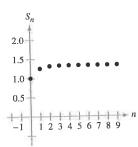**(d)** $S_n$



**49.** $g(1)$

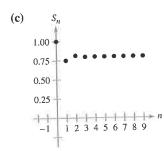**50.** $g(2)$

**51.** $g(3.1)$

**52.** $g(-2)$

*Writing*  In Exercises 53–56, match the graph of the first 10 terms of the sequence of partial sums of the series
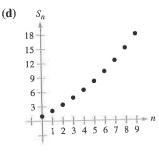
$$g(x) = \sum_{n=0}^{\infty} (2x)^n$$

with the indicated value of the function. [The graphs are labeled (a), (b), (c), and (d).] Explain how you made your choice.

**(a)** $S_n$



**(b)** $S_n$



**(c)** $S_n$



**(d)** $S_n$



**53.** $g\left(\dfrac{1}{8}\right)$

**54.** $g\left(-\dfrac{1}{8}\right)$

**55.** $g\left(\dfrac{9}{16}\right)$

**56.** $g\left(\dfrac{3}{8}\right)$

## Writing About Concepts

**57.** Define a power series centered at $c$.

**58.** Describe the radius of convergence of a power series. Describe the interval of convergence of a power series.

**59.** Describe the three basic forms of the domain of a power series.

## Writing About Concepts *(continued)*

**60.** Describe how to differentiate and integrate a power series with a radius of convergence $R$. Will the series resulting from the operations of differentiation and integration have a different radius of convergence? Explain.

**61.** Give examples that show that the convergence of a power series at an endpoint of its interval of convergence may be either conditional or absolute. Explain your reasoning.

**62.** Write a power series that has the indicated interval of convergence. Explain your reasoning.

    (a) $(-2, 2)$   (b) $(-1, 1]$   (c) $(-1, 0)$   (d) $[-2, 6)$

**63.** Let $f(x) = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $g(x) = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}$.

    (a) Find the intervals of convergence of $f$ and $g$.

    (b) Show that $f'(x) = g(x)$.

    (c) Show that $g'(x) = -f(x)$.

    (d) Identify the functions $f$ and $g$.

**64.** Let $f(x) = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$.

    (a) Find the interval of convergence of $f$.

    (b) Show that $f'(x) = f(x)$.

    (c) Show that $f(0) = 1$.

    (d) Identify the function $f$.

In Exercises 65–70, show that the function represented by the power series is a solution of the differential equation.

**65.** $y = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$,  $y'' + y = 0$

**66.** $y = \displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{(2n)!}$,  $y'' + y = 0$

**67.** $y = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}$,  $y'' - y = 0$

**68.** $y = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!}$,  $y'' - y = 0$
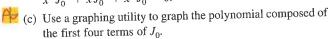
**69.** $y = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^{2n}}{2^n\, n!}$,  $y'' - xy' - y = 0$

**70.** $y = 1 + \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n x^{4n}}{2^{2n}\, n! \cdot 3 \cdot 7 \cdot 11 \cdots (4n-1)}$,  $y'' + x^2 y = 0$

**71.** *Bessel Function*  The Bessel function of order 0 is

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k}\, (k!)^2}.$$

    (a) Show that the series converges for all $x$.

    (b) Show that the series is a solution of the differential equation $x^2 J_0'' + x J_0' + x^2 J_0 = 0$.

    (c) Use a graphing utility to graph the polynomial composed of the first four terms of $J_0$.

    (d) Approximate $\int_0^1 J_0 \, dx$ accurate to two decimal places.

**72. Bessel Function**   The Bessel function of order 1 is

$$J_1(x) = x \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k+1} k!(k+1)!}.$$

(a) Show that the series converges for all $x$.

(b) Show that the series is a solution of the differential equation
$x^2 J_1'' + x J_1' + (x^2 - 1) J_1 = 0.$

(c) Use a graphing utility to graph the polynomial composed of the first four terms of $J_1$.

(d) Show that $J_0'(x) = -J_1(x)$.

In Exercises 73–76, the series represents a well-known function. Use a computer algebra system to graph the partial sum $S_{10}$ and identify the function from the graph.

**73.** $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$     **74.** $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

**75.** $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n, \quad -1 < x < 1$

**76.** $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad -1 \le x \le 1$

**77. Investigation**   In Exercise 11 you found that the interval of convergence of the geometric series $\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ is $(-2, 2)$.

(a) Find the sum of the series when $x = \frac{3}{4}$. Use a graphing utility to graph the first six terms of the sequence of partial sums and the horizontal line representing the sum of the series.

(b) Repeat part (a) for $x = -\frac{3}{4}$.

(c) Write a short paragraph comparing the rate of convergence of the partial sums with the sum of the series in parts (a) and (b). How do the plots of the partial sums differ as they converge toward the sum of the series?

(d) Given any positive real number $M$, there exists a positive integer $N$ such that the partial sum

$$\sum_{n=0}^{N} \left(\frac{3}{2}\right)^n > M.$$

Use a graphing utility to complete the table.

| M | 10 | 100 | 1000 | 10,000 |
|---|----|-----|------|--------|
| N |    |     |      |        |

**78. Investigation**   The interval of convergence of the series $\sum_{n=0}^{\infty} (3x)^n$ is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

(a) Find the sum of the series when $x = \frac{1}{6}$. Use a graphing utility to graph the first six terms of the sequence of partial sums and the horizontal line representing the sum of the series.

(b) Repeat part (a) for $x = -\frac{1}{6}$.

(c) Write a short paragraph comparing the rate of convergence of the partial sums with the sum of the series in parts (a) and (b). How do the plots of the partial sums differ as they converge toward the sum of the series?

(d) Given any positive real number $M$, there exists a positive integer $N$ such that the partial sum

$$\sum_{n=0}^{N} \left(3 \cdot \frac{2}{3}\right)^n > M.$$

Use a graphing utility to complete the table.

| M | 10 | 100 | 1000 | 10,000 |
|---|----|-----|------|--------|
| N |    |     |      |        |

**True or False?**   In Exercises 79–82, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

**79.** If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = 2$, then it also converges for $x = -2$.

**80.** If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = 2$, then it also converges for $x = -1$.

**81.** If the interval of convergence for $\sum_{n=0}^{\infty} a_n x^n$ is $(-1, 1)$, then the interval of convergence for $\sum_{n=0}^{\infty} a_n (x-1)^n$ is $(0, 2)$.

**82.** If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < 2$, then

$$\int_0^1 f(x)\, dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}.$$

**83.** Prove that the power series

$$\sum_{n=0}^{\infty} \frac{(n+p)!}{n!(n+q)!} x^n$$

has a radius of convergence of $R = \infty$ if $p$ and $q$ are positive integers.

**84.** Let $g(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$, where the coefficients are $c_{2n} = 1$ and $c_{2n+1} = 2$ for $n \ge 0$.

(a) Find the interval of convergence of the series.

(b) Find an explicit formula for $g(x)$.

**85.** Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_{n+3} = c_n$ for $n \ge 0$.

(a) Find the interval of convergence of the series.

(b) Find an explicit formula for $f(x)$.

**86.** Prove that if the power series $\sum_{n=0}^{\infty} c_n x^n$ has a radius of convergence of $R$, then $\sum_{n=0}^{\infty} c_n x^{2n}$ has a radius of convergence of $\sqrt{R}$.

**87.** For $n > 0$, let $R > 0$ and $c_n > 0$. Prove that if the interval of convergence of the series $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ is $(x_0 - R, x_0 + R]$, then the series converges conditionally at $x_0 + R$.

**Section 9.9**

## Representation of Functions by Power Series

- Find a geometric power series that represents a function.
- Construct a power series using series operations.

**JOSEPH FOURIER (1768–1830)**

Some of the early work in representing functions by power series was done by the French mathematician Joseph Fourier. Fourier's work is important in the history of calculus, partly because it forced eighteenth century mathematicians to question the then-prevailing narrow concept of a function. Both Cauchy and Dirichlet were motivated by Fourier's work with series, and in 1837 Dirichlet published the general definition of a function that is used today.

### Geometric Power Series

In this section and the next, you will study several techniques for finding a power series that represents a given function.

Consider the function given by $f(x) = 1/(1 - x)$. The form of $f$ closely resembles the sum of a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}, \quad |r| < 1.$$

In other words, if you let $a = 1$ and $r = x$, a power series representation for $1/(1 - x)$, centered at 0, is

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$$

$$= 1 + x + x^2 + x^3 + \cdots, \quad |x| < 1.$$

Of course, this series represents $f(x) = 1/(1 - x)$ only on the interval $(-1, 1)$, whereas $f$ is defined for all $x \neq 1$, as shown in Figure 9.22. To represent $f$ in another interval, you must develop a different series. For instance, to obtain the power series centered at $-1$, you could write

$$\frac{1}{1 - x} = \frac{1}{2 - (x + 1)} = \frac{1/2}{1 - [(x + 1)/2]} = \frac{a}{1 - r}$$

which implies that $a = \frac{1}{2}$ and $r = (x + 1)/2$. So, for $|x + 1| < 2$, you have

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{x + 1}{2}\right)^n$$

$$= \frac{1}{2}\left[1 + \frac{(x + 1)}{2} + \frac{(x + 1)^2}{4} + \frac{(x + 1)^3}{8} + \cdots\right], \quad |x + 1| < 2$$
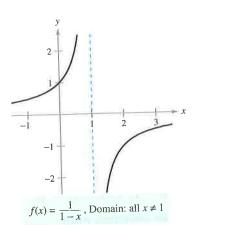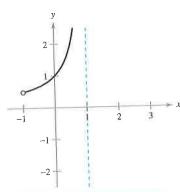
which converges on the interval $(-3, 1)$.



$f(x) = \dfrac{1}{1 - x}$, Domain: all $x \neq 1$

$f(x) = \displaystyle\sum_{n=0}^{\infty} x^n$, Domain: $-1 < x < 1$

**Figure 9.22**

### EXAMPLE 1 Finding a Geometric Power Series Centered at 0

Find a power series for $f(x) = \dfrac{4}{x + 2}$, centered at 0.

**Solution** Writing $f(x)$ in the form $a/(1 - r)$ produces

$$\frac{4}{2 + x} = \frac{2}{1 - (-x/2)} = \frac{a}{1 - r}$$

which implies that $a = 2$ and $r = -x/2$. So, the power series for $f(x)$ is

$$\frac{4}{x + 2} = \sum_{n=0}^{\infty} ar^n$$

$$= \sum_{n=0}^{\infty} 2\left(-\frac{x}{2}\right)^n$$

$$= 2\left(1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots\right).$$

This power series converges when

$$\left|-\frac{x}{2}\right| < 1$$

which implies that the interval of convergence is $(-2, 2)$.

*Long Division*

$$\begin{array}{r} 2 - x + \frac{1}{2}x^2 - \frac{1}{4}x^3 + \cdots \\ 2 + x \overline{)\,4\phantom{000000000000}} \\ 4 + 2x \\ \overline{-2x\phantom{00000}} \\ -2x - x^2 \\ \overline{x^2\phantom{0000}} \\ x^2 + \frac{1}{2}x^3 \\ \overline{-\frac{1}{2}x^3\phantom{00}} \\ -\frac{1}{2}x^3 - \frac{1}{4}x^4 \end{array}$$

Another way to determine a power series for a rational function such as the one in Example 1 is to use long division. For instance, by dividing $2 + x$ into 4, you obtain the result shown at the left.

### EXAMPLE 2 Finding a Geometric Power Series Centered at 1

Find a power series for $f(x) = \dfrac{1}{x}$, centered at 1.

**Solution** Writing $f(x)$ in the form $a/(1 - r)$ produces

$$\frac{1}{x} = \frac{1}{1 - (-x + 1)} = \frac{a}{1 - r}$$

which implies that $a = 1$ and $r = 1 - x = -(x - 1)$. So, the power series for $f(x)$ is

$$\frac{1}{x} = \sum_{n=0}^{\infty} ar^n$$

$$= \sum_{n=0}^{\infty} [-(x - 1)]^n$$

$$= \sum_{n=0}^{\infty} (-1)^n(x - 1)^n$$

$$= 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + \cdots.$$

This power series converges when

$$|x - 1| < 1$$

which implies that the interval of convergence is $(0, 2)$.

## Operations with Power Series

The versatility of geometric power series will be shown later in this section, following a discussion of power series operations. These operations, used with differentiation and integration, provide a means of developing power series for a variety of elementary functions. (For simplicity, the following properties are stated for a series centered at 0.)

---

**Operations with Power Series**

Let $f(x) = \Sigma a_n x^n$ and $g(x) = \Sigma b_n x^n$.

**1.** $f(kx) = \displaystyle\sum_{n=0}^{\infty} a_n k^n x^n$

**2.** $f(x^N) = \displaystyle\sum_{n=0}^{\infty} a_n x^{nN}$

**3.** $f(x) \pm g(x) = \displaystyle\sum_{n=0}^{\infty} (a_n \pm b_n) x^n$

---

The operations described above can change the interval of convergence for the resulting series. For example, in the following addition, the interval of convergence for the sum is the *intersection* of the intervals of convergence of the two original series.

$$\underbrace{\sum_{n=0}^{\infty} x^n}_{(-1,1)} + \underbrace{\sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n}_{\cap \ (-2,2)} = \underbrace{\sum_{n=0}^{\infty} \left(1 + \frac{1}{2^n}\right)x^n}_{= \quad (-1,1)}$$

### EXAMPLE 3    Adding Two Power Series

Find a power series, centered at 0, for $f(x) = \dfrac{3x - 1}{x^2 - 1}$.

**Solution**    Using partial fractions, you can write $f(x)$ as

$$\frac{3x - 1}{x^2 - 1} = \frac{2}{x + 1} + \frac{1}{x - 1}.$$

By adding the two geometric power series

$$\frac{2}{x + 1} = \frac{2}{1 - (-x)} = \sum_{n=0}^{\infty} 2(-1)^n x^n, \quad |x| < 1$$

and

$$\frac{1}{x - 1} = \frac{-1}{1 - x} = -\sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

you obtain the following power series.

$$\frac{3x - 1}{x^2 - 1} = \sum_{n=0}^{\infty} [2(-1)^n - 1]x^n = 1 - 3x + x^2 - 3x^3 + x^4 - \cdots$$

The interval of convergence for this power series is $(-1, 1)$.

## EXAMPLE 4 Finding a Power Series by Integration

Find a power series for $f(x) = \ln x$, centered at 1.

**Solution** From Example 2, you know that

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n.$$

Interval of convergence: (0, 2)

Integrating this series produces

$$\ln x = \int \frac{1}{x}\,dx + C$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}.$$

By letting $x = 1$, you can conclude that $C = 0$. Therefore,

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

$$= \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots.$$

Interval of convergence: (0, 2]

Note that the series converges at $x = 2$. This is consistent with the observation in the preceding section that integration of a power series may alter the convergence at the endpoints of the interval of convergence.

**TECHNOLOGY** In Section 9.7, the fourth-degree Taylor polynomial for the natural logarithmic function

$$\ln x \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

was used to approximate $\ln(1.1)$.

$$\ln(1.1) \approx (0.1) - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4$$

$$\approx 0.0953083$$

You now know from Example 4 that this polynomial represents the first four terms of the power series for $\ln x$. Moreover, using the Alternating Series Remainder, you can determine that the error in this approximation is less than

$$|R_4| \le |a_5|$$

$$= \frac{1}{5}(0.1)^5$$

$$= 0.000002.$$

During the seventeenth and eighteenth centuries, mathematical tables for logarithms and values of other transcendental functions were computed in this manner. Such numerical techniques are far from outdated, because it is precisely by such means that many modern calculating devices are programmed to evaluate transcendental functions.

## EXAMPLE 5   Finding a Power Series by Integration

Find a power series for $g(x) = \arctan x$, centered at 0.

**Solution**   Because $D_x[\arctan x] = 1/(1 + x^2)$, you can use the series

$$f(x) = \frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

<span style="color:red">Interval of convergence: $(-1, 1)$</span>

Substituting $x^2$ for $x$ produces

$$f(x^2) = \frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

Finally, by integrating, you obtain

$$\arctan x = \int \frac{1}{1 + x^2}\, dx + C$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1}$$

<span style="color:red">Let $x = 0$, then $C = 0$.</span>

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

<span style="color:red">Interval of convergence: $(-1, 1)$</span>

It can be shown that the power series developed for $\arctan x$ in Example 5 also converges (to $\arctan x$) for $x = \pm 1$. For instance, when $x = 1$, you can write

$$\arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$= \frac{\pi}{4}.$$

However, this series (developed by James Gregory in 1671) does not give us a practical way of approximating $\pi$ because it converges so slowly that hundreds of terms would have to be used to obtain reasonable accuracy. Example 6 shows how to use *two* different arctangent series to obtain a very good approximation of $\pi$ using only a few terms. This approximation was developed by John Machin in 1706.

## EXAMPLE 6   Approximating $\pi$ with a Series

Use the trigonometric identity

$$4 \arctan \frac{1}{5} - \arctan \frac{1}{239} = \frac{\pi}{4}$$

to approximate the number $\pi$ [see Exercise 50(b)].

**Solution**   By using only five terms from each of the series for $\arctan(1/5)$ and $\arctan(1/239)$, you obtain

$$4\left(4 \arctan \frac{1}{5} - \arctan \frac{1}{239}\right) \approx 3.1415926$$

which agrees with the exact value of $\pi$ with an error of less than 0.0000001.

---

**Srinivasa Ramanujan (1887–1920)**

Series that can be used to approximate $\pi$ have interested mathematicians for the past 300 years. An amazing series for approximating $1/\pi$ was discovered by the Indian mathematician Srinivasa Ramanujan in 1914 (see Exercise 64). Each successive term of Ramanujan's series adds roughly eight more correct digits to the value of $1/\pi$. For more information about Ramanujan's work, see the article "Ramanujan and Pi" by Jonathan M. Borwein and Peter B. Borwein in *Scientific American*.

## Exercises for Section 9.9

In Exercises 1–4, find a geometric power series for the function, centered at 0, (a) by the technique shown in Examples 1 and 2 and (b) by long division.

**1.** $f(x) = \dfrac{1}{2 - x}$

**2.** $f(x) = \dfrac{4}{5 - x}$

**3.** $f(x) = \dfrac{1}{2 + x}$

**4.** $f(x) = \dfrac{1}{1 + x}$

In Exercises 5–16, find a power series for the function, centered at c, and determine the interval of convergence.

**5.** $f(x) = \dfrac{1}{2 - x}, \quad c = 5$

**6.** $f(x) = \dfrac{4}{5 - x}, \quad c = -2$

**7.** $f(x) = \dfrac{3}{2x - 1}, \quad c = 0$

**8.** $f(x) = \dfrac{3}{2x - 1}, \quad c = 2$

**9.** $g(x) = \dfrac{1}{2x - 5}, \quad c = -3$

**10.** $h(x) = \dfrac{1}{2x - 5}, \quad c = 0$

**11.** $f(x) = \dfrac{3}{x + 2}, \quad c = 0$

**12.** $f(x) = \dfrac{4}{3x + 2}, \quad c = 2$

**13.** $g(x) = \dfrac{3x}{x^2 + x - 2}, \quad c = 0$

**14.** $g(x) = \dfrac{4x - 7}{2x^2 + 3x - 2}, \quad c = 0$

**15.** $f(x) = \dfrac{2}{1 - x^2}, \quad c = 0$

**16.** $f(x) = \dfrac{4}{4 + x^2}, \quad c = 0$

In Exercises 17–26, use the power series

$$\frac{1}{1 + x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

to determine a power series, centered at 0, for the function. Identify the interval of convergence.

**17.** $h(x) = \dfrac{-2}{x^2 - 1} = \dfrac{1}{1 + x} + \dfrac{1}{1 - x}$

**18.** $h(x) = \dfrac{x}{x^2 - 1} = \dfrac{1}{2(1 + x)} - \dfrac{1}{2(1 - x)}$

**19.** $f(x) = -\dfrac{1}{(x + 1)^2} = \dfrac{d}{dx}\left[\dfrac{1}{x + 1}\right]$

**20.** $f(x) = \dfrac{2}{(x + 1)^3} = \dfrac{d^2}{dx^2}\left[\dfrac{1}{x + 1}\right]$

**21.** $f(x) = \ln(x + 1) = \displaystyle\int \dfrac{1}{x + 1}\, dx$

**22.** $f(x) = \ln(1 - x^2) = \displaystyle\int \dfrac{1}{1 + x}\, dx - \int \dfrac{1}{1 - x}\, dx$

**23.** $g(x) = \dfrac{1}{x^2 + 1}$

**24.** $f(x) = \ln(x^2 + 1)$

**25.** $h(x) = \dfrac{1}{4x^2 + 1}$

**26.** $f(x) = \arctan 2x$

**Graphical and Numerical Analysis** In Exercises 27 and 28, let

$$S_n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \pm \frac{x^n}{n}.$$

Use a graphing utility to confirm the inequality graphically. Then complete the table to confirm the inequality numerically.

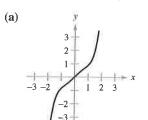| $x$ | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $S_n$ | | | | | | |
| $\ln(x + 1)$ | | | | | | |
| $S_{n+1}$ | | | | | | |

**27.** $S_2 \le \ln(x + 1) \le S_3$
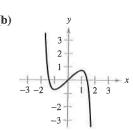**28.** $S_4 \le \ln(x + 1) \le S_5$

In Exercises 29 and 30, (a) graph several partial sums of the series, (b) find the sum of the series and its radius of convergence, (c) use 50 terms of the series to approximate the sum when $x = 0.5$, and (d) determine what the approximation represents and how good the approximation is.
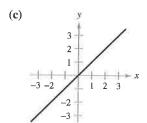
**29.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x - 1)^n}{n}$
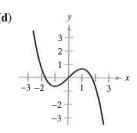
**30.** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n + 1)!}$

In Exercises 31–34, match the polynomial approximation of the function $f(x) = \arctan x$ with the correct graph. [The graphs are labeled (a), (b), (c), and (d).]

(a)



(b)



(c)



(d)



**31.** $g(x) = x$

**32.** $g(x) = x - \dfrac{x^3}{3}$

**33.** $g(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5}$

**34.** $g(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7}$

In Exercises 35–38, use the series for $f(x) = \arctan x$ to approximate the value, using $R_N \le 0.001$.

**35.** $\arctan \dfrac{1}{4}$

**36.** $\displaystyle\int_0^{3/4} \arctan x^2 \, dx$

**37.** $\displaystyle\int_0^{1/2} \dfrac{\arctan x^2}{x} \, dx$

**38.** $\displaystyle\int_0^{1/2} x^2 \arctan x \, dx$

In Exercises 39–42, use the power series

$$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

Find the series representation of the function and determine its interval of convergence.

**39.** $f(x) = \dfrac{1}{(1-x)^2}$

**40.** $f(x) = \dfrac{x}{(1-x)^2}$

**41.** $f(x) = \dfrac{1+x}{(1-x)^2}$

**42.** $f(x) = \dfrac{x(1+x)}{(1-x)^2}$

**43.** *Probability*   A fair coin is tossed repeatedly. The probability that the first head occurs on the $n$th toss is $P(n) = \left(\frac{1}{2}\right)^n$. When this game is repeated many times, the average number of tosses required until the first head occurs is

$$E(n) = \sum_{n=1}^{\infty} nP(n).$$

(This value is called the *expected value of n*.) Use the results of Exercises 39–42 to find $E(n)$. Is the answer what you expected? Why or why not?

**44.** Use the results of Exercises 39–42 to find the sum of each series.

(a) $\dfrac{1}{3}\displaystyle\sum_{n=1}^{\infty} n\left(\dfrac{2}{3}\right)^n$

(b) $\dfrac{1}{10}\displaystyle\sum_{n=1}^{\infty} n\left(\dfrac{9}{10}\right)^n$

*Writing*   In Exercises 45–48, explain how to use the geometric series

$$g(x) = \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

to find the series for the function. Do not find the series.

**45.** $f(x) = \dfrac{1}{1+x}$

**46.** $f(x) = \dfrac{1}{1-x^2}$

**47.** $f(x) = \dfrac{5}{1+x}$

**48.** $f(x) = \ln(1-x)$

**49.** Prove that $\arctan x + \arctan y = \arctan \dfrac{x+y}{1-xy}$ for $xy \ne 1$ provided the value of the left side of the equation is between $-\pi/2$ and $\pi/2$.

**50.** Use the result of Exercise 49 to verify each identity.

(a) $\arctan \dfrac{120}{119} - \arctan \dfrac{1}{239} = \dfrac{\pi}{4}$

(b) $4 \arctan \dfrac{1}{5} - \arctan \dfrac{1}{239} = \dfrac{\pi}{4}$

[*Hint:* Use Exercise 49 twice to find $4 \arctan \frac{1}{5}$. Then use part (a).]

In Exercises 51 and 52, (a) verify the given equation and (b) use the equation and the series for the arctangent to approximate $\pi$ to two-decimal-place accuracy.

**51.** $2 \arctan \dfrac{1}{2} - \arctan \dfrac{1}{7} = \dfrac{\pi}{4}$

**52.** $\arctan \dfrac{1}{2} + \arctan \dfrac{1}{3} = \dfrac{\pi}{4}$

In Exercises 53–58, find the sum of the convergent series by using a well-known function. Identify the function and explain how you obtained the sum.

**53.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{2^n n}$

**54.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{3^n n}$

**55.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{2^n}{5^n n}$

**56.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{1}{2n+1}$

**57.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n \dfrac{1}{2^{2n+1}(2n+1)}$

**58.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{1}{3^{2n-1}(2n-1)}$

## Writing About Concepts

**59.** Use the results of Exercises 31–34 to make a geometric argument for why the series approximations of $f(x) = \arctan x$ have only odd powers of $x$.

**60.** Use the results of Exercises 31–34 to make a conjecture about the degrees of series approximations of $f(x) = \arctan x$ that have relative extrema.

**61.** One of the series in Exercises 53–58 converges to its sum at a much lower rate than the other five series. Which is it? Explain why this series converges so slowly. Use a graphing utility to illustrate the rate of convergence.

**62.** The radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is 3. What is the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} na_n x^{n-1}$? Explain.

**63.** The power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ converges for $|x + 1| < 4$. What can you conclude about the series $\displaystyle\sum_{n=0}^{\infty} a_n \dfrac{x^{n+1}}{n+1}$? Explain.

**64.** Use a graphing utility to show that

$$\dfrac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \dfrac{(4n)!(1103 + 26{,}390n)}{(n!)^4 396^{4n}} = \dfrac{1}{\pi}.$$

(*Note:* This series was discovered by the Indian mathematician Srinivasa Ramanujan in 1914.)

In Exercises 65 and 66, find the sum of the series.

**65.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n}{3^n(2n+1)}$

**66.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{(-1)^n \pi^{2n+1}}{3^{2n+1}(2n+1)!}$

**Section 9.10**

# Taylor and Maclaurin Series

- Find a Taylor or Maclaurin series for a function.
- Find a binomial series.
- Use a basic list of Taylor series to find other Taylor series.

## Taylor Series and Maclaurin Series

In Section 9.9, you derived power series for several functions using geometric series with term-by-term differentiation or integration. In this section you will study a *general* procedure for deriving the power series for a function that has derivatives of all orders. The following theorem gives the form that *every* convergent power series must take.

**COLIN MACLAURIN (1698–1746)**

The development of power series to represent functions is credited to the combined work of many seventeenth and eighteenth century mathematicians. Gregory, Newton, John and James Bernoulli, Leibniz, Euler, Lagrange, Wallis, and Fourier all contributed to this work. However, the two names that are most commonly associated with power series are Brook Taylor (1685–1731) and Colin Maclaurin.

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**THEOREM 9.22   The Form of a Convergent Power Series**

If $f$ is represented by a power series $f(x) = \sum a_n(x - c)^n$ for all $x$ in an open interval $I$ containing $c$, then $a_n = f^{(n)}(c)/n!$ and

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots.$$

---

**Proof**   Suppose the power series $\sum a_n(x - c)^n$ has a radius of convergence $R$. Then, by Theorem 9.21, you know that the $n$th derivative of $f$ exists for $|x - c| < R$, and by successive differentiation you obtain the following.

$$f^{(0)}(x) = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + a_4(x - c)^4 + \cdots$$
$$f^{(1)}(x) = a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + 4a_4(x - c)^3 + \cdots$$
$$f^{(2)}(x) = 2a_2 + 3!a_3(x - c) + 4 \cdot 3a_4(x - c)^2 + \cdots$$
$$f^{(3)}(x) = 3!a_3 + 4!a_4(x - c) + \cdots$$
$$\vdots$$
$$f^{(n)}(x) = n!a_n + (n + 1)!a_{n+1}(x - c) + \cdots$$

Evaluating each of these derivatives at $x = c$ yields

$$f^{(0)}(c) = 0!a_0$$
$$f^{(1)}(c) = 1!a_1$$
$$f^{(2)}(c) = 2!a_2$$
$$f^{(3)}(c) = 3!a_3$$

and, in general, $f^{(n)}(c) = n!a_n$. By solving for $a_n$, you find that the coefficients of the power series representation of $f(x)$ are

$$a_n = \frac{f^{(n)}(c)}{n!}.$$

**NOTE**   Be sure you understand Theorem 9.22. The theorem says that *if a power series converges to $f(x)$*, the series must be a Taylor series. The theorem does *not* say that every series formed with the Taylor coefficients $a_n = f^{(n)}(c)/n!$ will converge to $f(x)$.

Notice that the coefficients of the power series in Theorem 9.22 are precisely the coefficients of the Taylor polynomials for $f(x)$ at $c$ as defined in Section 9.7. For this reason, the series is called the **Taylor series** for $f(x)$ at $c$.

---

### Definitions of Taylor and Maclaurin Series

If a function $f$ has derivatives of all orders at $x = c$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n = f(c) + f'(c)(x - c) + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

is called the **Taylor series for $f(x)$ at $c$.** Moreover, if $c = 0$, then the series is the **Maclaurin series for $f$.**

---

If you know the pattern for the coefficients of the Taylor polynomials for a function, you can extend the pattern easily to form the corresponding Taylor series. For instance, in Example 4 in Section 9.7, you found the fourth Taylor polynomial for $\ln x$, centered at 1, to be

$$P_4(x) = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4.$$

From this pattern, you can obtain the Taylor series for $\ln x$ centered at $c = 1$,

$$(x - 1) - \frac{1}{2}(x - 1)^2 + \cdots + \frac{(-1)^{n+1}}{n}(x - 1)^n + \cdots.$$

### EXAMPLE 1    Forming a Power Series

Use the function $f(x) = \sin x$ to form the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

and determine the interval of convergence.

**Solution**    Successive differentiation of $f(x)$ yields

$$\begin{array}{ll}
f(x) = \sin x & f(0) = \sin 0 = 0 \\
f'(x) = \cos x & f'(0) = \cos 0 = 1 \\
f''(x) = -\sin x & f''(0) = -\sin 0 = 0 \\
f^{(3)}(x) = -\cos x & f^{(3)}(0) = -\cos 0 = -1 \\
f^{(4)}(x) = \sin x & f^{(4)}(0) = \sin 0 = 0 \\
f^{(5)}(x) = \cos x & f^{(5)}(0) = \cos 0 = 1
\end{array}$$

and so on. The pattern repeats after the third derivative. So, the power series is as follows.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = 0 + (1)x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6$$

$$+ \frac{(-1)}{7!}x^7 + \cdots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

By the Ratio Test, you can conclude that this series converges for all $x$.
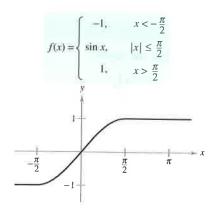
$$f(x) = \begin{cases} -1, & x < -\frac{\pi}{2} \\ \sin x, & |x| \le \frac{\pi}{2} \\ 1, & x > \frac{\pi}{2} \end{cases}$$



**Figure 9.23**

Notice that in Example 1 you cannot conclude that the power series converges to $\sin x$ for all $x$. You can simply conclude that the power series converges to some function, but you are not sure what function it is. This is a subtle, but important, point in dealing with Taylor or Maclaurin series. To persuade yourself that the series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

might converge to a function other than $f$, remember that the derivatives are being evaluated at a single point. It can easily happen that another function will agree with the values of $f^{(n)}(x)$ when $x = c$ and disagree at other $x$-values. For instance, if you formed the power series (centered at 0) for the function shown in Figure 9.23, you would obtain the same series as in Example 1. You know that the series converges for all $x$, and yet it obviously cannot converge to both $f(x)$ and $\sin x$ for all $x$.

Let $f$ have derivatives of all orders in an open interval $I$ centered at $c$. The Taylor series for $f$ may fail to converge for some $x$ in $I$. Or, even if it is convergent, it may fail to have $f(x)$ as its sum. Nevertheless, Theorem 9.19 tells us that for each $n$,

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}.$$

Note that in this remainder formula the particular value of $z$ that makes the remainder formula true depends on the values of $x$ and $n$. If $R_n \to 0$, then the following theorem tells us that the Taylor series for $f$ actually converges to $f(x)$ for all $x$ in $I$.

---

**THEOREM 9.23    Convergence of Taylor Series**

If $\lim_{n \to \infty} R_n = 0$ for all $x$ in the interval $I$, then the Taylor series for $f$ converges and equals $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x - c)^n.$$

---

**Proof**   For a Taylor series, the $n$th partial sum coincides with the $n$th Taylor polynomial. That is, $S_n(x) = P_n(x)$. Moreover, because

$$P_n(x) = f(x) - R_n(x)$$

it follows that

$$\begin{aligned} \lim_{n \to \infty} S_n(x) &= \lim_{n \to \infty} P_n(x) \\ &= \lim_{n \to \infty} [f(x) - R_n(x)] \\ &= f(x) - \lim_{n \to \infty} R_n(x). \end{aligned}$$

So, for a given $x$, the Taylor series (the sequence of partial sums) converges to $f(x)$ if and only if $R_n(x) \to 0$ as $n \to \infty$.

**NOTE**    Stated another way, Theorem 9.23 says that a power series formed with Taylor coefficients $a_n = f^{(n)}(c)/n!$ converges to the function from which it was derived at precisely those values for which the remainder approaches 0 as $n \to \infty$.

In Example 1, you derived the power series from the sine function and you also concluded that the series converges to some function on the entire real line. In Example 2, you will see that the series actually converges to $\sin x$. The key observation is that although the value of $z$ is not known, it is possible to obtain an upper bound for $\left| f^{(n+1)}(z) \right|$.

## EXAMPLE 2    A Convergent Maclaurin Series

Show that the Maclaurin series for $f(x) = \sin x$ converges to $\sin x$ for all $x$.

**Solution**    Using the result in Example 1, you need to show that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdot \cdot \cdot + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdot \cdot \cdot$$

is true for all $x$. Because

$$f^{(n+1)}(x) = \pm \sin x$$

or

$$f^{(n+1)}(x) = \pm \cos x$$

you know that $\left| f^{(n+1)}(z) \right| \leq 1$ for every real number $z$. Therefore, for any fixed $x$, you can apply Taylor's Theorem (Theorem 9.19) to conclude that
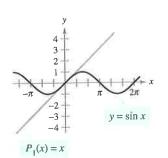
$$0 \leq |R_n(x)| = \left| \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}.$$
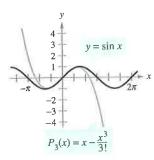
From the discussion in Section 9.1 regarding the relative rates of convergence of exponential and factorial sequences, it follows that for a fixed $x$
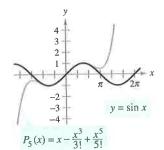
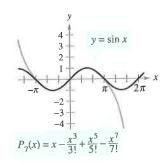$$\lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0.$$

Finally, by the Squeeze Theorem, it follows that for all $x$, $R_n(x) \to 0$ as $n \to \infty$. So, by Theorem 9.23, the Maclaurin series for $\sin x$ converges to $\sin x$ for all $x$.

Figure 9.24 visually illustrates the convergence of the Maclaurin series for $\sin x$ by comparing the graphs of the Maclaurin polynomials $P_1(x)$, $P_3(x)$, $P_5(x)$, and $P_7(x)$ with the graph of the sine function. Notice that as the degree of the polynomial increases, its graph more closely resembles that of the sine function.



$$P_1(x) = x$$

$$P_3(x) = x - \frac{x^3}{3!}$$

$$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

As $n$ increases, the graph of $P_n$ more closely resembles the sine function.
**Figure 9.24**

The guidelines for finding a Taylor series for $f(x)$ at $c$ are summarized below.

**Guidelines for Finding a Taylor Series**

1. Differentiate $f(x)$ several times and evaluate each derivative at $c$.
   $$f(c), f'(c), f''(c), f'''(c), \cdots, f^{(n)}(c), \cdots$$
   Try to recognize a pattern in these numbers.

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$, and determine the interval of convergence for the resulting power series
   $$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots.$$

3. Within this interval of convergence, determine whether or not the series converges to $f(x)$.

The direct determination of Taylor or Maclaurin coefficients using successive differentiation can be difficult, and the next example illustrates a shortcut for finding the coefficients indirectly—using the coefficients of a known Taylor or Maclaurin series.

### EXAMPLE 3 Maclaurin Series for a Composite Function

Find the Maclaurin series for $f(x) = \sin x^2$.

**Solution** To find the coefficients for this Maclaurin series directly, you must calculate successive derivatives of $f(x) = \sin x^2$. By calculating just the first two,

$$f'(x) = 2x \cos x^2 \quad \text{and} \quad f''(x) = -4x^2 \sin x^2 + 2 \cos x^2$$

you can see that this task would be quite cumbersome. Fortunately, there is an alternative. First consider the Maclaurin series for $\sin x$ found in Example 1.

$$g(x) = \sin x$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Now, because $\sin x^2 = g(x^2)$, you can substitute $x^2$ for $x$ in the series for $\sin x$ to obtain

$$\sin x^2 = g(x^2)$$
$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots.$$

Be sure to understand the point illustrated in Example 3. Because direct computation of Taylor or Maclaurin coefficients can be tedious, the most practical way to find a Taylor or Maclaurin series is to develop power series for a *basic list* of elementary functions. From this list, you can determine power series for other functions by the operations of addition, subtraction, multiplication, division, differentiation, integration, or composition with known power series.

## Binomial Series

Before presenting the basic list for elementary functions, you wll develop one more series—for a function of the form $f(x) = (1 + x)^k$. This produces the **binomial series.**

### EXAMPLE 4    Binomial Series

Find the Maclaurin series for $f(x) = (1 + x)^k$ and determine its radius of convergence. Assume that $k$ is not a positive integer.

**Solution**    By successive differentiation, you have

$$f(x) = (1 + x)^k \qquad\qquad f(0) = 1$$
$$f'(x) = k(1 + x)^{k-1} \qquad\qquad f'(0) = k$$
$$f''(x) = k(k - 1)(1 + x)^{k-2} \qquad\qquad f''(0) = k(k - 1)$$
$$f'''(x) = k(k - 1)(k - 2)(1 + x)^{k-3} \qquad f'''(0) = k(k - 1)(k - 2)$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$f^{(n)}(x) = k \cdot \cdots (k - n + 1)(1 + x)^{k-n} \qquad f^{(n)}(0) = k(k - 1) \cdots (k - n + 1)$$

which produces the series

$$1 + kx + \frac{k(k - 1)x^2}{2} + \cdots + \frac{k(k - 1) \cdots (k - n + 1)x^n}{n!} + \cdots .$$

Because $a_{n+1}/a_n \to 1$, you can apply the Ratio Test to conclude that the radius of convergence is $R = 1$. So, the series converges to some function in the interval $(-1, 1)$.

Note that Example 4 shows that the Taylor series for $(1 + x)^k$ converges to some function in the interval $(-1, 1)$. However, the example does not show that the series actually converges to $(1 + x)^k$. To do this, you could show that the remainder $R_n(x)$ converges to 0, as illustrated in Example 2.

### EXAMPLE 5    Finding a Binomial Series

Find the power series for $f(x) = \sqrt[3]{1 + x}$.

**Solution**    Using the binomial series

$$(1 + x)^k = 1 + kx + \frac{k(k - 1)x^2}{2!} + \frac{k(k - 1)(k - 2)x^3}{3!} + \cdots$$

let $k = \frac{1}{3}$ and write

$$(1 + x)^{1/3} = 1 + \frac{x}{3} - \frac{2x^2}{3^2 2!} + \frac{2 \cdot 5x^3}{3^3 3!} - \frac{2 \cdot 5 \cdot 8x^4}{3^4 4!} + \cdots$$

which converges for $-1 \le x \le 1$.

**TECHNOLOGY**    Use a graphing utility to confirm the result in Example 5. When you graph the functions

$$f(x) = (1 + x)^{1/3} \quad \text{and} \quad P_4(x) = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5x^3}{81} - \frac{10x^4}{243}$$

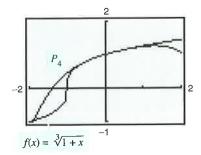in the same viewing window, you should obtain the result shown in Figure 9.25.



$f(x) = \sqrt[3]{1 + x}$

**Figure 9.25**

## Deriving Taylor Series from a Basic List

The following list provides the power series for several elementary functions with the corresponding intervals of convergence.

**Power Series for Elementary Functions**

| Function | Interval of Convergence |
|---|---|
| $\dfrac{1}{x} = 1 - (x - 1) + (x - 1)^2 - (x - 1)^3 + (x - 1)^4 - \cdots + (-1)^n (x - 1)^n + \cdots$ | $0 < x < 2$ |
| $\dfrac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$ | $-1 < x < 1$ |
| $\ln x = (x - 1) - \dfrac{(x - 1)^2}{2} + \dfrac{(x - 1)^3}{3} - \dfrac{(x - 1)^4}{4} + \cdots + \dfrac{(-1)^{n-1}(x - 1)^n}{n} + \cdots$ | $0 < x \le 2$ |
| $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \cdots + \dfrac{x^n}{n!} + \cdots$ | $-\infty < x < \infty$ |
| $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \cdots + \dfrac{(-1)^n x^{2n+1}}{(2n + 1)!} + \cdots$ | $-\infty < x < \infty$ |
| $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \cdots + \dfrac{(-1)^n x^{2n}}{(2n)!} + \cdots$ | $-\infty < x < \infty$ |
| $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \cdots + \dfrac{(-1)^n x^{2n+1}}{2n + 1} + \cdots$ | $-1 \le x \le 1$ |
| $\arcsin x = x + \dfrac{x^3}{2 \cdot 3} + \dfrac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \dfrac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \dfrac{(2n)!x^{2n+1}}{(2^n n!)^2(2n + 1)} + \cdots$ | $-1 \le x \le 1$ |
| $(1 + x)^k = 1 + kx + \dfrac{k(k - 1)x^2}{2!} + \dfrac{k(k - 1)(k - 2)x^3}{3!} + \dfrac{k(k - 1)(k - 2)(k - 3)x^4}{4!} + \cdots$ | $-1 < x < 1*$ |

\* *The convergence at $x = \pm 1$ depends on the value of $k$.*

NOTE  The binomial series is valid for noninteger values of $k$. Moreover, if $k$ happens to be a positive integer, the binomial series reduces to a simple binomial expansion.

### EXAMPLE 6  Deriving a Power Series from a Basic List

Find the power series for $f(x) = \cos \sqrt{x}$.

**Solution**  Using the power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

you can replace $x$ by $\sqrt{x}$ to obtain the series

$$\cos \sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges for all $x$ in the domain of $\cos \sqrt{x}$—that is, for $x \ge 0$.

Power series can be multiplied and divided like polynomials. After finding the first few terms of the product (or quotient), you may be able to recognize a pattern.

## EXAMPLE 7  Multiplication and Division of Power Series

Find the first three nonzero terms in each of the Maclaurin series.

**a.** $e^x \arctan x$    **b.** $\tan x$

### Solution

**a.** Using the Maclaurin series for $e^x$ and $\arctan x$ in the table, you have

$$e^x \arctan x = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right).$$

Multiply these expressions and collect like terms as you would for multiplying polynomials.

$$
\begin{aligned}
&1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \cdots \\[4pt]
&\underline{\quad x \qquad\quad -\tfrac{1}{3}x^3 \qquad\quad + \tfrac{1}{5}x^5 - \cdots} \\[4pt]
&x + \ x^2 + \tfrac{1}{2}x^3 + \ \tfrac{1}{6}x^4 + \tfrac{1}{24}x^5 + \cdots \\[4pt]
&\qquad\qquad\quad -\tfrac{1}{3}x^3 - \ \tfrac{1}{3}x^4 - \ \tfrac{1}{6}x^5 - \cdots \\[4pt]
&\underline{\qquad\qquad\qquad\qquad\qquad\qquad + \ \tfrac{1}{5}x^5 + \cdots} \\[4pt]
&x + \ x^2 + \tfrac{1}{6}x^3 - \ \tfrac{1}{6}x^4 + \tfrac{3}{40}x^5 + \cdots
\end{aligned}
$$

So, $e^x \arctan x = x + x^2 + \tfrac{1}{6}x^3 + \cdots$.

**b.** Using the Maclaurin series for $\sin x$ and $\cos x$ in the table, you have

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots}{1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots}.$$

Divide using long division.

$$
\begin{array}{r}
x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots \\[4pt]
1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \cdots\ \overline{\big)\ x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \cdots} \\[4pt]
\underline{x - \tfrac{1}{2}x^3 + \tfrac{1}{24}x^5 - \cdots} \\[4pt]
\tfrac{1}{3}x^3 - \tfrac{1}{30}x^5 + \cdots \\[4pt]
\underline{\tfrac{1}{3}x^3 - \tfrac{1}{6}x^5 + \cdots} \\[4pt]
\tfrac{2}{15}x^5 + \cdots
\end{array}
$$

So, $\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$.

### EXAMPLE 8 A Power Series for $\sin^2 x$

Find the power series for $f(x) = \sin^2 x$.

**Solution** Consider rewriting $\sin^2 x$ as follows.

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2} - \frac{\cos 2x}{2}$$

Now, use the series for $\cos x$.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\cos 2x = 1 - \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 - \frac{2^6}{6!}x^6 + \frac{2^8}{8!}x^8 - \cdots$$

$$-\frac{1}{2}\cos 2x = -\frac{1}{2} + \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \cdots$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x = \frac{1}{2} - \frac{1}{2} + \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \cdots$$

$$= \frac{2}{2!}x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 - \frac{2^7}{8!}x^8 + \cdots$$

This series converges for $-\infty < x < \infty$.

As mentioned in the preceding section, power series can be used to obtain tables of values of transcendental functions. They are also useful for estimating the values of definite integrals for which antiderivatives cannot be found. The next example demonstrates this use.

### EXAMPLE 9 Power Series Approximation of a Definite Integral

Use a power series to approximate

$$\int_0^1 e^{-x^2}\,dx$$

with an error of less than 0.01.

**Solution** Replacing $x$ with $-x^2$ in the series for $e^x$ produces the following.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

$$\int_0^1 e^{-x^2}\,dx = \left[ x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \cdots \right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots$$

Summing the first four terms, you have

$$\int_0^1 e^{-x^2}\,dx \approx 0.74$$

which, by the Alternating Series Test, has an error of less than $\frac{1}{216} \approx 0.005$.

In Exercises 1–10, use the definition to find the Taylor series (centered at $c$) for the function.

**1.** $f(x) = e^{2x}$,  $c = 0$

**2.** $f(x) = e^{3x}$,  $c = 0$

**3.** $f(x) = \cos x$,  $c = \dfrac{\pi}{4}$

**4.** $f(x) = \sin x$,  $c = \dfrac{\pi}{4}$

**5.** $f(x) = \ln x$,  $c = 1$

**6.** $f(x) = e^x$,  $c = 1$

**7.** $f(x) = \sin 2x$,  $c = 0$

**8.** $f(x) = \ln(x^2 + 1)$,  $c = 0$

**9.** $f(x) = \sec x$,  $c = 0$ (first three nonzero terms)

**10.** $f(x) = \tan x$,  $c = 0$ (first three nonzero terms)

In Exercises 11–14, prove that the Maclaurin series for the function converges to the function for all $x$.

**11.** $f(x) = \cos x$     **12.** $f(x) = e^{-2x}$

**13.** $f(x) = \sinh x$     **14.** $f(x) = \cosh x$

In Exercises 15–20, use the binomial series to find the Maclaurin series for the function.

**15.** $f(x) = \dfrac{1}{(1 + x)^2}$

**16.** $f(x) = \dfrac{1}{\sqrt{1 - x}}$

**17.** $f(x) = \dfrac{1}{\sqrt{4 + x^2}}$

**18.** $f(x) = \sqrt[4]{1 + x}$

**19.** $f(x) = \sqrt{1 + x^2}$

**20.** $f(x) = \sqrt{1 + x^3}$

In Exercises 21–30, find the Maclaurin series for the function. (Use the table of power series for elementary functions.)

**21.** $f(x) = e^{x^2/2}$

**22.** $g(x) = e^{-3x}$

**23.** $g(x) = \sin 3x$

**24.** $f(x) = \cos 4x$

**25.** $f(x) = \cos x^{3/2}$

**26.** $g(x) = 2 \sin x^3$

**27.** $f(x) = \frac{1}{2}(e^x - e^{-x}) = \sinh x$

**28.** $f(x) = e^x + e^{-x} = 2 \cosh x$

**29.** $f(x) = \cos^2 x$

**30.** $f(x) = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$

$\left(\text{\emph{Hint:} Integrate the series for } \dfrac{1}{\sqrt{x^2 + 1}}.\right)$

In Exercises 31–34, find the Maclaurin series for the function. (See Example 7.)

**31.** $f(x) = x \sin x$     **32.** $h(x) = x \cos x$

**33.** $g(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$    **34.** $f(x) = \begin{cases} \dfrac{\arcsin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

In Exercises 35 and 36, use a power series and the fact that $i^2 = -1$ to verify the formula.

**35.** $g(x) = \dfrac{1}{2i}(e^{ix} - e^{-ix}) = \sin x$

**36.** $g(x) = \frac{1}{2}(e^{ix} + e^{-ix}) = \cos x$

In Exercises 37–42, find the first four nonzero terms of the Maclaurin series for the function by multiplying or dividing the appropriate power series. Use the table of power series for elementary functions on page 682. Use a graphing utility to graph the function and its corresponding polynomial approximation.

**37.** $f(x) = e^x \sin x$     **38.** $g(x) = e^x \cos x$
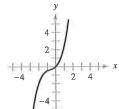
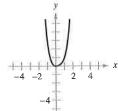**39.** $h(x) = \cos x \ln(1 + x)$     **40.** $f(x) = e^x \ln(1 + x)$

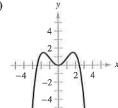**41.** $g(x) = \dfrac{\sin x}{1 + x}$     **42.** $f(x) = \dfrac{e^x}{1 + x}$

In Exercises 43–46, match the polynomial with its graph. [The graphs are labeled (a), (b), (c), and (d).] Factor a common factor from each polynomial and identify the function approximated by the remaining Taylor polynomial.
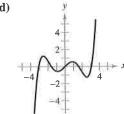
**(a)**



**(b)**



**(c)**



**(d)**



**43.** $y = x^2 - \dfrac{x^4}{3!}$     **44.** $y = x - \dfrac{x^3}{2!} + \dfrac{x^5}{4!}$
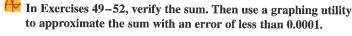
**45.** $y = x + x^2 + \dfrac{x^3}{2!}$     **46.** $y = x^2 - x^3 + x^4$

**In Exercises 47 and 48, find a Maclaurin series for $f(x)$.**

**47.** $f(x) = \int_0^x (e^{-t^2} - 1)\, dt$

**48.** $f(x) = \int_0^x \sqrt{1 + t^3}\, dt$

**In Exercises 49–52, verify the sum. Then use a graphing utility to approximate the sum with an error of less than 0.0001.**

**49.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln 2$

**50.** $\displaystyle\sum_{n=0}^{\infty} (-1)^n \left[ \frac{1}{(2n+1)!} \right] = \sin 1$

**51.** $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$

**52.** $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{1}{n!} \right) = \frac{e-1}{e}$

**In Exercises 53 and 54, use the series representation of the function $f$ to find $\displaystyle\lim_{x \to 0} f(x)$ (if it exists).**

**53.** $f(x) = \dfrac{1 - \cos x}{x}$

**54.** $f(x) = \dfrac{\sin x}{x}$

**In Exercises 55–58, use a power series to approximate the value of the integral with an error of less than 0.0001. (In Exercises 55 and 56, assume that the integrand is defined as 1 when $x = 0$.)**
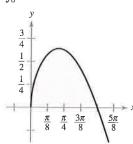
**55.** $\displaystyle\int_0^1 \frac{\sin x}{x}\, dx$

**56.** $\displaystyle\int_0^{1/2} \frac{\arctan x}{x}\, dx$

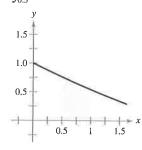**57.** $\displaystyle\int_{0.1}^{0.3} \sqrt{1 + x^3}\, dx$

**58.** $\displaystyle\int_0^{1/4} x \ln(x + 1)\, dx$

**Area    In Exercises 59 and 60, use a power series to approximate the area of the region. Use a graphing utility to verify the result.**

**59.** $\displaystyle\int_0^{\pi/2} \sqrt{x} \cos x\, dx$     **60.** $\displaystyle\int_{0.5}^{1} \cos \sqrt{x}\, dx$
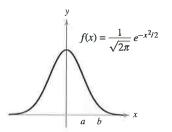


*Probability*   **In Exercises 61 and 62, approximate the normal probability with an error of less than 0.0001, where the probability is given by**

$$P(a < x < b) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx.$$


$f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$

**61.** $P(0 < x < 1)$

**62.** $P(1 < x < 2)$

**In Exercises 63–66, use a computer algebra system to find the fifth-degree Taylor polynomial (centered at $c$) for the function. Graph the function and the polynomial. Use the graph to determine the largest interval on which the polynomial is a reasonable approximation of the function.**

**63.** $f(x) = x \cos 2x, \quad c = 0$

**64.** $f(x) = \sin \dfrac{x}{2} \ln(1 + x), \quad c = 0$

**65.** $g(x) = \sqrt{x} \ln x, \quad c = 1$

**66.** $h(x) = \sqrt[3]{x} \arctan x, \quad c = 1$

## Writing About Concepts

**67.** State the guidelines for finding a Taylor series.

**68.** If $f$ is an even function, what must be true about the coefficients $a_n$ in the Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n?$$

Explain your reasoning.

**69.** Explain how to use the series

$$g(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

to find the series for each function. Do not find the series.

(a) $f(x) = e^{-x}$

(b) $f(x) = e^{3x}$

(c) $f(x) = xe^x$

(d) $f(x) = e^{2x} + e^{-2x}$

**70.** Define the binomial series. What is its radius of convergence?

**71. *Projectile Motion***   A projectile fired from the ground follows the trajectory given by

$$y = \left(\tan\theta - \frac{g}{kv_0\cos\theta}\right)x - \frac{g}{k^2}\ln\left(1 - \frac{kx}{v_0\cos\theta}\right)$$

where $v_0$ is the initial speed, $\theta$ is the angle of projection, $g$ is the acceleration due to gravity, and $k$ is the drag factor caused by air resistance. Using the power series representation

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad -1 < x < 1$$

verify that the trajectory can be rewritten as

$$y = (\tan\theta)x + \frac{gx^2}{2v_0^2\cos^2\theta} + \frac{kgx^3}{3v_0^3\cos^3\theta} + \frac{k^2 gx^4}{4v_0^4\cos^4\theta} + \cdots.$$

**72. *Projectile Motion***   Use the result of Exercise 71 to determine the series for the path of a projectile launched from ground level at an angle of $\theta = 60°$, with an initial speed of $v_0 = 64$ feet per second and a drag factor of $k = \frac{1}{16}$.

**73. *Investigation***   Consider the function $f$ defined by

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

(a) Sketch a graph of the function.

(b) Use the alternative form of the definition of the derivative (Section 2.1) and L'Hôpital's Rule to show that $f'(0) = 0$. [By continuing this process, it can be shown that $f^{(n)}(0) = 0$ for $n > 1$.]

(c) Using the result in part (b), find the Maclaurin series for $f$. Does the series converge to $f$?

**74. *Investigation***

(a) Find the power series centered at 0 for the function

$$f(x) = \frac{\ln(x^2 + 1)}{x^2}.$$

(b) Use a graphing utility to graph $f$ and the eighth-degree Taylor polynomial $P_8(x)$ for $f$.

(c) Complete the table, where

$$F(x) = \int_0^x \frac{\ln(t^2 + 1)}{t^2}\,dt \quad \text{and} \quad G(x) = \int_0^x P_8(t)\,dt.$$

| $x$ | 0.25 | 0.50 | 0.75 | 1.00 | 1.50 | 2.00 |
|---|---|---|---|---|---|---|
| $F(x)$ | | | | | | |
| $G(x)$ | | | | | | |

(d) Describe the relationship between the graphs of $f$ and $P_8$ and the results given in the table in part (c).

**75.** Prove that $\displaystyle\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any real $x$.

**76.** Find the Maclaurin series for

$$f(x) = \ln\frac{1 + x}{1 - x}$$

and determine its radius of convergence. Use the first four terms of the series to approximate $\ln 3$.

**In Exercises 77–80, evaluate the binomial coefficient using the formula**

$$\binom{k}{n} = \frac{k(k-1)(k-2)(k-3)\cdots(k-n+1)}{n!}$$

**where $k$ is a real number, $n$ is a positive integer, and**

$$\binom{k}{0} = 1.$$

**77.** $\dbinom{5}{3}$

**78.** $\dbinom{-2}{2}$

**79.** $\dbinom{0.5}{4}$

**80.** $\dbinom{-1/3}{5}$

**81.** Write the power series for $(1 + x)^k$ in terms of binomial coefficients.

**82.** Prove that $e$ is irrational. $\Big[$Hint: Assume that $e = p/q$ is rational ($p$ and $q$ are integers) and consider

$$e = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} + \cdots.\Big]$$

**83.** Show that the Maclaurin series of the function

$$g(x) = \frac{x}{1 - x - x^2}$$

is

$$\sum_{n=1}^{\infty} F_n x^n$$

where $F_n$ is the $n$th Fibonacci number with $F_1 = F_2 = 1$ and $F_n = F_{n-2} + F_{n-1}$, for $n \geq 3$.

(*Hint:* Write

$$\frac{x}{1 - x - x^2} = a_0 + a_1 x + a_2 x^2 + \cdots$$

and multiply each side of this equation by $1 - x - x^2$.)

### Putnam Exam Challenge

**84.** Assume that $|f(x)| \leq 1$ and $|f''(x)| \leq 1$ for all $x$ on an interval of length at least 2. Show that $|f'(x)| \leq 2$ on the interval.
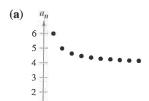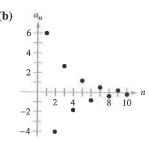
In Exercises 1 and 2, write an expression for the $n$th term of the sequence.
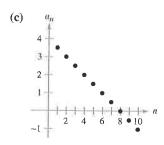
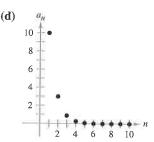**1.** $1, \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{24}, \dfrac{1}{120}, \cdots$

**2.** $\dfrac{1}{2}, \dfrac{2}{5}, \dfrac{3}{10}, \dfrac{4}{17}, \cdots$

In Exercises 3–6, match the sequence with its graph. [The graphs are labeled (a), (b), (c), and (d).]

**(a)**



**(b)**



**(c)**



**(d)**



**3.** $a_n = 4 + \dfrac{2}{n}$

**4.** $a_n = 4 - \dfrac{1}{2}n$

**5.** $a_n = 10(0.3)^{n-1}$

**6.** $a_n = 6\left(-\dfrac{2}{3}\right)^{n-1}$

In Exercises 7 and 8, use a graphing utility to graph the first 10 terms of the sequence. Use the graph to make an inference about the convergence or divergence of the sequence. Verify your inference analytically and, if the sequence converges, find its limit.

**7.** $a_n = \dfrac{5n + 2}{n}$

**8.** $a_n = \sin \dfrac{n\pi}{2}$

In Exercises 9–16, determine the convergence or divergence of the sequence with the given $n$th term. If the sequence converges, find its limit. ($b$ and $c$ are positive real numbers.)

**9.** $a_n = \dfrac{n + 1}{n^2}$

**10.** $a_n = \dfrac{1}{\sqrt{n}}$

**11.** $a_n = \dfrac{n^3}{n^2 + 1}$

**12.** $a_n = \dfrac{n}{\ln n}$

**13.** $a_n = \sqrt{n + 1} - \sqrt{n}$

**14.** $a_n = \left(1 + \dfrac{1}{2n}\right)^n$

**15.** $a_n = \dfrac{\sin \sqrt{n}}{\sqrt{n}}$

**16.** $a_n = (b^n + c^n)^{1/n}$

**17.** *Compound Interest* A deposit of \$5000 is made in an account that earns 5% interest compounded quarterly. The balance in the account after $n$ quarters is

$$A_n = 5000\left(1 + \dfrac{0.05}{4}\right)^n, \quad n = 1, 2, 3, \cdots.$$

(a) Compute the first eight terms of the sequence $\{A_n\}$.

(b) Find the balance in the account after 10 years by computing the 40th term of the sequence.

**18.** *Depreciation* A company buys a machine for \$120,000. During the next 5 years the machine will depreciate at a rate of 30% per year. (That is, at the end of each year, the depreciated value will be 70% of what it was at the beginning of the year.)

(a) Find a formula for the $n$th term of the sequence that gives the value $V$ of the machine $t$ full years after it was purchased.

(b) Find the depreciated value of the machine at the end of 5 full years.

*Numerical, Graphical, and Analytic Analysis* In Exercises 19–22, (a) use a graphing utility to find the indicated partial sum $S_k$ and complete the table, and (b) use a graphing utility to graph the first 10 terms of the sequence of partial sums.

| $k$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $S_k$ | | | | | |

**19.** $\displaystyle\sum_{n=1}^{\infty} \left(\dfrac{3}{2}\right)^{n-1}$

**20.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{2n}$

**21.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{(2n)!}$

**22.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n(n + 1)}$

In Exercises 23–26, determine the convergence or divergence of the series.

**23.** $\displaystyle\sum_{n=0}^{\infty} (0.82)^n$

**24.** $\displaystyle\sum_{n=0}^{\infty} (1.82)^n$

**25.** $\displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n n}{\ln n}$

**26.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{2n + 1}{3n + 2}$

In Exercises 27–30, find the sum of the convergent series.

**27.** $\displaystyle\sum_{n=0}^{\infty} \left(\dfrac{2}{3}\right)^n$

**28.** $\displaystyle\sum_{n=0}^{\infty} \dfrac{2^{n+2}}{3^n}$

**29.** $\displaystyle\sum_{n=0}^{\infty} \left(\dfrac{1}{2^n} - \dfrac{1}{3^n}\right)$

**30.** $\displaystyle\sum_{n=0}^{\infty} \left[\left(\dfrac{2}{3}\right)^n - \dfrac{1}{(n + 1)(n + 2)}\right]$

In Exercises 31 and 32, (a) write the repeating decimal as a geometric series and (b) write its sum as the ratio of two integers.

**31.** $0.\overline{09}$

**32.** $0.\overline{923076}$

**33.** *Distance*   A ball is dropped from a height of 8 meters. Each time it drops $h$ meters, it rebounds $0.7h$ meters. Find the total distance traveled by the ball.

**34.** *Salary*   You accept a job that pays a salary of \$32,000 the first year. During the next 39 years, you will receive a 5.5% raise each year. What would be your total compensation over the 40-year period?

**35.** *Compound Interest*   A deposit of \$200 is made at the end of each month for 2 years in an account that pays 6% interest, compounded continuously. Determine the balance in the account at the end of 2 years.

**36.** *Compound Interest*   A deposit of \$100 is made at the end of each month for 10 years in an account that pays 3.5%, compounded monthly. Determine the balance in the account at the end of 10 years.

In Exercises 37–40, determine the convergence or divergence of the series.

**37.** $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^4}$

**38.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}}$

**39.** $\displaystyle\sum_{n=1}^{\infty} \left( \frac{1}{n^2} - \frac{1}{n} \right)$

**40.** $\displaystyle\sum_{n=1}^{\infty} \left( \frac{1}{n^2} - \frac{1}{2^n} \right)$

In Exercises 41–44, determine the convergence or divergence of the series.

**41.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 2n}}$

**42.** $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}$

**43.** $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$

**44.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n - 5}$

In Exercises 45–48, determine the convergence or divergence of the series.

**45.** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2 - 3}$

**46.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+1}$

**47.** $\displaystyle\sum_{n=4}^{\infty} \frac{(-1)^n n}{n-3}$

**48.** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n \ln n^3}{n}$

In Exercises 49–52, determine the convergence or divergence of the series.

**49.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$

**50.** $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^n}$

**51.** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$

**52.** $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 5 \cdot 8 \cdots (3n-1)}$

*Numerical, Graphical, and Analytic Analysis*   In Exercises 53 and 54, (a) verify that the series converges, (b) use a graphing utility to find the indicated partial sum $S_n$ and complete the table, (c) use a graphing utility to graph the first 10 terms of the sequence of partial sums, and (d) use the table to estimate the sum of the series.

| $n$ | 5 | 10 | 15 | 20 | 25 |
|-----|---|----|----|----|----|
| $S_n$ | | | | | |

**53.** $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{3}{5}\right)^n$

**54.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{n^3 + 5}$

**55.** *Writing*   Use a graphing utility to complete the table for (a) $p = 2$ and (b) $p = 5$. Write a short paragraph describing and comparing the entries in the table.

| $N$ | 5 | 10 | 20 | 30 | 40 |
|-----|---|----|----|----|----|
| $\displaystyle\sum_{n=1}^{N} \frac{1}{n^p}$ | | | | | |
| $\displaystyle\int_{N}^{\infty} \frac{1}{x^p}\, dx$ | | | | | |

**56.** *Writing*   You are told that the terms of a positive series appear to approach zero very slowly as $n$ approaches infinity. (In fact, $a_{75} = 0.7$.) If you are given no other information, can you conclude that the series diverges? Support your answer with an example.

In Exercises 57 and 58, find the third-degree Taylor polynomial centered at $c$.

**57.** $f(x) = e^{-x/2}, \quad c = 0$

**58.** $f(x) = \tan x, \quad c = -\dfrac{\pi}{4}$

In Exercises 59–62, use a Taylor polynomial to approximate the function with an error of less than 0.001.

**59.** $\sin 95°$

**60.** $\cos(0.75)$

**61.** $\ln(1.75)$

**62.** $e^{-0.25}$

**63.** A Taylor polynomial centered at 0 will be used to approximate the cosine function. Find the degree of the polynomial required to obtain the desired accuracy over each interval.

| Maximum Error | Interval |
|---------------|----------|
| (a) 0.001 | $[-0.5, 0.5]$ |
| (b) 0.001 | $[-1, 1]$ |
| (c) 0.0001 | $[-0.5, 0.5]$ |
| (d) 0.0001 | $[-2, 2]$ |

**64.** Use a graphing utility to graph the cosine function and the Taylor polynomials in Exercise 63.

**In Exercises 65–70, find the interval of convergence of the power series. (Be sure to include a check for convergence at the endpoints of the interval.)**

65. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{10}\right)^n$

66. $\displaystyle\sum_{n=0}^{\infty} (2x)^n$

67. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{(n+1)^2}$

68. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n (x-2)^n}{n}$

69. $\displaystyle\sum_{n=0}^{\infty} n!(x-2)^n$

70. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{2^n}$

**In Exercises 71 and 72, show that the function represented by the power series is a solution of the differential equation.**

71. $y = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n (n!)^2}$

$x^2 y'' + x y' + x^2 y = 0$

72. $y = \displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{2^n n!}$

$y'' + 3xy' + 3y = 0$

**In Exercises 73 and 74, find a geometric power series centered at 0 for the function.**

73. $g(x) = \dfrac{2}{3-x}$

74. $h(x) = \dfrac{3}{2+x}$

75. Find a power series for the derivative of the function in Exercise 73.

76. Find a power series for the integral of the function in Exercise 74.

**In Exercises 77 and 78, find a function represented by the series and give the domain of the function.**

77. $1 + \dfrac{2}{3}x + \dfrac{4}{9}x^2 + \dfrac{8}{27}x^3 + \cdots$

78. $8 - 2(x-3) + \dfrac{1}{2}(x-3)^2 - \dfrac{1}{8}(x-3)^3 + \cdots$

**In Exercises 79–86, find a power series for the function centered at c.**

79. $f(x) = \sin x, \quad c = \dfrac{3\pi}{4}$

80. $f(x) = \cos x, \quad c = -\dfrac{\pi}{4}$

81. $f(x) = 3^x, \quad c = 0$

82. $f(x) = \csc x, \quad c = \dfrac{\pi}{2}$

(first three terms)

83. $f(x) = \dfrac{1}{x}, \quad c = -1$

84. $f(x) = \sqrt{x}, \quad c = 4$

85. $g(x) = \sqrt[5]{1+x}, \quad c = 0$

86. $h(x) = \dfrac{1}{(1+x)^3}, \quad c = 0$

**In Exercises 87–92, find the sum of the convergent series by using a well-known function. Identify the function and explain how you obtained the sum.**

87. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{4^n n}$

88. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{5^n n}$

89. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n n!}$

90. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^n n!}$

91. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{3^{2n} (2n)!}$

92. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1} (2n+1)!}$

93. *Writing*  One of the series in Exercises 41 and 49 converges to its sum at a much lower rate than the other series. Which is it? Explain why this series converges so slowly. Use a graphing utility to illustrate the rate of convergence.

94. Use the binomial series to find the Maclaurin series for

$f(x) = \dfrac{1}{\sqrt{1+x^3}}.$

95. *Forming Maclaurin Series*  Determine the first four terms of the Maclaurin series for $e^{2x}$

(a) by using the definition of the Maclaurin series and the formula for the coefficient of the $n$th term, $a_n = f^{(n)}(0)/n!.$

(b) by replacing $x$ by $2x$ in the series for $e^x$.

(c) by multiplying the series for $e^x$ by itself, because $e^{2x} = e^x \cdot e^x.$

96. *Forming Maclaurin Series*  Follow the pattern of Exercise 95 to find the first four terms of the series for $\sin 2x$. (*Hint:* $\sin 2x = 2 \sin x \cos x$.)

**In Exercises 97–100, find the series representation of the function defined by the integral.**

97. $\displaystyle\int_0^x \frac{\sin t}{t}\, dt$

98. $\displaystyle\int_0^x \cos \frac{\sqrt{t}}{2}\, dt$

99. $\displaystyle\int_0^x \frac{\ln(t+1)}{t}\, dt$

100. $\displaystyle\int_0^x \frac{e^t - 1}{t}\, dt$

**In Exercises 101 and 102, use a power series to find the limit (if it exists). Verify the result by using L'Hôpital's Rule.**
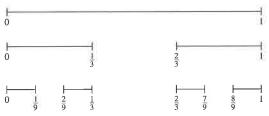
101. $\displaystyle\lim_{x \to 0^+} \frac{\arctan x}{\sqrt{x}}$

102. $\displaystyle\lim_{x \to 0} \frac{\arcsin x}{x}$
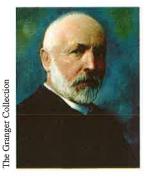
**1.** The Cantor set (Georg Cantor, 1845–1918) is a subset of the unit interval $[0, 1]$. To construct the Cantor set, first remove the middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$ of the interval, leaving two line segments. For the second step, remove the middle third of each of the two remaining segments, leaving four line segments. Continue this procedure indefinitely, as shown in the figure. The Cantor set consists of all numbers in the unit interval $[0, 1]$ that still remain.



(a) Find the total length of all the line segments that are removed.

(b) Write down three numbers that are in the Cantor set.

(c) Let $C_n$ denote the total length of the remaining line segments after $n$ steps. Find $\lim\limits_{n \to \infty} C_n$.



**GEORG CANTOR (1845–1918)**

Cantor was a German mathematician known for his work on the development of set theory, which is the basis of modern mathematical analysis. This theory extends to the concept of infinite (or transfinite) numbers.

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**2.** It can be shown that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{[see Example 3(b), Section 9.3].}$$

Use this fact to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.
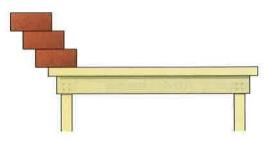
**3.** Let $T$ be an equilateral triangle with sides of length 1. Let $a_n$ be the number of circles that can be packed tightly in $n$ rows inside the triangle. For example, $a_1 = 1$, $a_2 = 3$, and $a_3 = 6$, as shown in the figure. Let $A_n$ be the combined area of the $a_n$ circles. Find $\lim\limits_{n \to \infty} A_n$.



**4.** Identical blocks of unit length are stacked on top of each other at the edge of a table. The center of gravity of the top block must lie over the block below it, the center of gravity of the top two blocks must lie over the block below them, and so on (see figure).



(a) If there are three blocks, show that it is possible to stack them so that the left edge of the top block extends $\frac{11}{12}$ unit beyond the edge of the table.

(b) Is it possible to stack the blocks so that the right edge of the top block extends beyond the edge of the table?

(c) How far beyond the table can the blocks be stacked?

**5.** (a) Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n = 1 + 2x + 3x^2 + x^3 + 2x^4 + 3x^5 + x^6 + \cdots$$

in which the coefficients $a_n = 1, 2, 3, 1, 2, 3, 1, \ldots$ are periodic of period $p = 3$. Find the radius of convergence and the sum of this power series.

(b) Consider a power series

$$\sum_{n=0}^{\infty} a_n x^n$$

in which the coefficients are periodic, $(a_{n+p} = a_p)$ and $a_n > 0$. Find the radius of convergence and the sum of this power series.

**6.** For what values of the positive constants $a$ and $b$ does the following series converge absolutely? For what values does it converge conditionally?

$$a - \frac{b}{2} + \frac{a}{3} - \frac{b}{4} + \frac{a}{5} - \frac{b}{6} + \frac{a}{7} - \frac{b}{8} + \cdots$$

**7.** (a) Find a power series for the function

$$f(x) = xe^x$$

centered at 0. Use this representation to find the sum of the infinite series

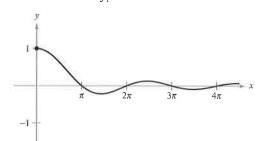$$\sum_{n=1}^{\infty} \frac{1}{n!(n+2)}.$$

(b) Differentiate the power series for $f(x) = xe^x$. Use the result to find the sum of the infinite series

$$\sum_{n=0}^{\infty} \frac{n+1}{n!}.$$

8. Find $f^{(12)}(0)$ if $f(x) = e^{x^2}$. (*Hint:* Do not calculate 12 derivatives.)

9. The graph of the function

$$f(x) = \begin{cases} 1, & x = 0 \\ \dfrac{\sin x}{x}, & x > 0 \end{cases}$$

is shown below. Use the Alternating Series Test to show that the improper integral $\displaystyle\int_1^\infty f(x)\, dx$ converges.



10. (a) Prove that $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^p}\, dx$ converges if and only if $p > 1$.

    (b) Determine the convergence or divergence of the series

    $$\sum_{n=4}^\infty \frac{1}{n \ln(n^2)}.$$

11. (a) Consider the following sequence of numbers defined recursively.

    $$a_1 = 3$$
    $$a_2 = \sqrt{3}$$
    $$a_3 = \sqrt{3 + \sqrt{3}}$$
    $$\vdots$$
    $$a_{n+1} = \sqrt{3 + a_n}$$

    Write the decimal approximations for the first six terms of this sequence. Prove that the sequence converges and find its limit.

    (b) Consider the following sequence defined recursively by $a_1 = \sqrt{a}$ and $a_{n+1} = \sqrt{a + a_n}$, where $a > 2$.

    $$\sqrt{a},\ \sqrt{a + \sqrt{a}},\ \sqrt{a + \sqrt{a + \sqrt{a}}},\ \dots$$

    Prove that this sequence converges and find its limit.

12. Let $\{a_n\}$ be a sequence of positive numbers satisfying $\displaystyle\lim_{n\to\infty} (a_n)^{1/n} = L < \frac{1}{r}$, $r > 0$. Prove that the series $\displaystyle\sum_{n=1}^\infty a_n r^n$ converges.

13. Consider the infinite series $\displaystyle\sum_{n=1}^\infty \frac{1}{2^{n+(-1)^n}}$.

    (a) Find the first five terms of the sequence of partial sums.

    (b) Show that the Ratio Test is inconclusive for this series.

    (c) Use the Root Test to test for the convergence or divergence of this series.

14. Derive each identity using the appropriate geometric series.

    (a) $\dfrac{1}{0.99} = 1.01010101\dots$    (b) $\dfrac{1}{0.98} = 1.0204081632\dots$

15. Consider an idealized population with the characteristic that each member of the population produces one offspring at the end of every time period. Each member has a life span of three time periods and the population begins with 10 newborn members. The following table shows the population during the first five time periods.
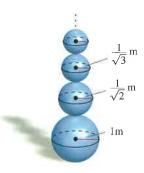
|  | Time Period | | | | |
|---|---|---|---|---|---|
| Age Bracket | 1 | 2 | 3 | 4 | 5 |
| 0–1 | 10 | 10 | 20 | 40 | 70 |
| 1–2 | | 10 | 10 | 20 | 40 |
| 2–3 | | | 10 | 10 | 20 |
| Total | 10 | 20 | 40 | 70 | 130 |

The sequence for the total population has the property that

$$S_n = S_{n-1} + S_{n-2} + S_{n-3}, \qquad n > 3.$$

Find the total population during each of the next five time periods.

16. Imagine you are stacking an infinite number of spheres of decreasing radii on top of each other, as shown in the figure. The radii of the spheres are 1 meter, $1/\sqrt{2}$ meter, $1/\sqrt{3}$ meter, etc. The spheres are made of a material that weighs 1 newton per cubic meter.

    (a) How high is this infinite stack of spheres?

    (b) What is the total surface area of all the spheres in the stack?

    (c) Show that the weight of the stack is finite.



17. (a) Determine the convergence or divergence of the series

    $$\sum_{n=1}^\infty \frac{1}{2n}.$$

    (b) Determine the convergence or divergence of the series

    $$\sum_{n=1}^\infty \left( \sin \frac{1}{2n} - \sin \frac{1}{2n+1} \right).$$