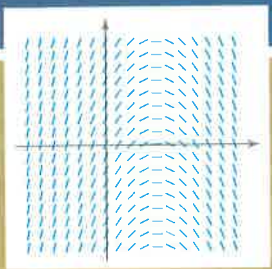
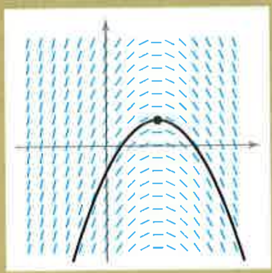
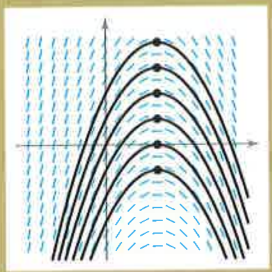
6 Differential Equations







A function $y = f(x)$ is a solution of a differential equation if the equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives. One way to solve a differential equation is to use slope fields, which show the general shape of all solutions of a differential equation. In Chapter 6, you will learn how to sketch slope fields and solve differential equations.

The SkyDome in Toronto is an entertainment center that has a retractable roof. How do you think the noise levels in the stadium compare when the roof is open and when the roof is closed? Explain your reasoning.

Image Gap/Alamy Images

Section 6.1 Slope Fields and Euler's Method

- **Use initial conditions to find particular solutions of differential equations.**
- **Use slope fields to approximate solutions of differential equations.**
- **Use Euler's Method to approximate solutions of differential equations.**

General and Particular Solutions

In this text, you will learn that physical phenomena can be described by differential equations. In Section 6.2, you will see that problems involving radioactive decay, population growth, and Newton's Law of Cooling can be formulated in terms of differential equations.

A function $y = f(x)$ is called a **solution** of a differential equation if the equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives. For example, differentiation and substitution would show that $y = e^{-2x}$ is a solution of the differential equation $y' + 2y = 0$. It can be shown that every solution of this differential equation is of the form

$$y = Ce^{-2x}$$ General solution of $y' + 2y = 0$

where C is any real number. This solution is called the **general solution.** Some differential equations have **singular solutions** that cannot be written as special cases of the general solution. However, such solutions are not considered in this text. The **order** of a differential equation is determined by the highest-order derivative in the equation. For instance, $y' = 4y$ is a first-order differential equation.

NOTE First-order linear differential equations are discussed in Section 6.4.

In Section 4.1, Example 8, you saw that the second-order differential equation $s''(t) = -32$ has the general solution

$$s(t) = -16t^2 + C_1t + C_2$$ General solution of $s''(t) = -32$

which contains two arbitrary constants. It can be shown that a differential equation of order n has a general solution with n arbitrary constants.

EXAMPLE 1 Verifying Solutions

Determine whether the function is a solution of the differential equation $y'' - y = 0$.

a. $y = \sin x$ **b.** $y = 4e^{-x}$ **c.** $y = Ce^x$

Solution

a. Because $y = \sin x$, $y' = \cos x$, and $y'' = -\sin x$, it follows that

$$y'' - y = -\sin x - \sin x = -2\sin x \neq 0.$$

So, $y = \sin x$ is *not* a solution.

b. Because $y = 4e^{-x}$, $y' = -4e^{-x}$, and $y'' = 4e^{-x}$, it follows that

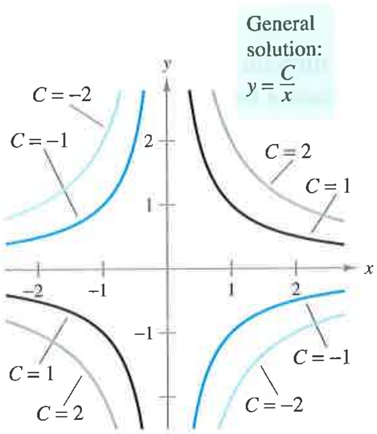
$$y'' - y = 4e^{-x} - 4e^{-x} = 0.$$

So, $y = 4e^{-x}$ is a solution.

c. Because $y = Ce^x$, $y' = Ce^x$, and $y'' = Ce^x$, it follows that

$$y'' - y = Ce^x - Ce^x = 0.$$

So, $y = Ce^x$ is a solution for any value of C.



Solution curves for $xy' + y = 0$
Figure 6.1

Geometrically, the general solution of a first-order differential equation represents a family of curves known as **solution curves,** one for each value assigned to the arbitrary constant. For instance, you can verify that every function of the form

$$y = \frac{C}{x} \qquad \text{General solution of } xy' + y = 0$$

is a solution of the differential equation $xy' + y = 0$. Figure 6.1 shows four of the solution curves corresponding to different values of C.

As discussed in Section 4.1, **particular solutions** of a differential equation are obtained from **initial conditions** that give the value of the dependent variable or one of its derivatives for a particular value of the independent variable. The term "initial condition" stems from the fact that, often in problems involving time, the value of the dependent variable or one of its derivatives is known at the *initial* time $t = 0$. For instance, the second-order differential equation $s''(t) = -32$ having the general solution

$$s(t) = -16t^2 + C_1t + C_2 \qquad \text{General solution of } s''(t) = -32$$

might have the following initial conditions.

$$s(0) = 80, \qquad s'(0) = 64 \qquad \text{Initial conditions}$$

In this case, the initial conditions yield the particular solution

$$s(t) = -16t^2 + 64t + 80. \qquad \text{Particular solution}$$

EXAMPLE 2 Finding a Particular Solution

For the differential equation $xy' - 3y = 0$, verify that $y = Cx^3$ is a solution, and find the particular solution determined by the initial condition $y = 2$ when $x = -3$.

Solution You know that $y = Cx^3$ is a solution because $y' = 3Cx^2$ and

$$\begin{aligned} xy' - 3y &= x(3Cx^2) - 3(Cx^3) \\ &= 0. \end{aligned}$$

Furthermore, the initial condition $y = 2$ when $x = -3$ yields

$$y = Cx^3 \qquad \text{General solution}$$

$$2 = C(-3)^3 \qquad \text{Substitute initial condition.}$$

$$-\frac{2}{27} = C \qquad \text{Solve for } C.$$

and you can conclude that the particular solution is

$$y = -\frac{2x^3}{27}. \qquad \text{Particular solution}$$

Try checking this solution by substituting for y and y' in the original differential equation.

NOTE To determine a particular solution, the number of initial conditions must match the number of constants in the general solution.

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indicates that in the HM mathSpace® *CD-ROM and the online* Eduspace® *system for this text, you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

Slope Fields

Solving a differential equation analytically can be difficult or even impossible. However, there is a graphical approach you can use to learn a lot about the solution of a differential equation. Consider a differential equation of the form

$$y' = F(x, y). \qquad \text{Differential equation}$$

At each point (x, y) in the xy-plane where F is defined, the differential equation determines the slope $y' = F(x, y)$ of the solution at that point. If you draw a short line segment with slope $F(x, y)$ at selected points (x, y) in the domain of F, then these line segments form a **slope field,** or a *direction field* for the differential equation $y' = F(x, y)$. Each line segment has the same slope as the solution curve through that point. A slope field shows the general shape of all the solutions.

EXAMPLE 3 Sketching a Slope Field

Sketch a slope field for the differential equation $y' = x - y$ for the points $(-1, 1)$, $(0, 1)$, and $(1, 1)$.

Solution

The slope of the solution curve at any point (x, y) is $F(x, y) = x - y$. So, the slope at $(-1, 1)$ is $y' = -1 - 1 = -2$, the slope at $(0, 1)$ is $y' = 0 - 1 = -1$, and the slope at $(1, 1)$ is $y' = 1 - 1 = 0$. Draw short line segments at the three points with their respective slopes, as shown in Figure 6.2.

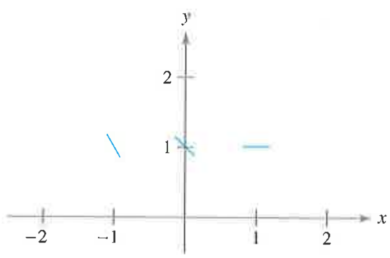


Figure 6.2

EXAMPLE 4 Identifying Slope Fields for Differential Equations

Match each slope field with its differential equation.

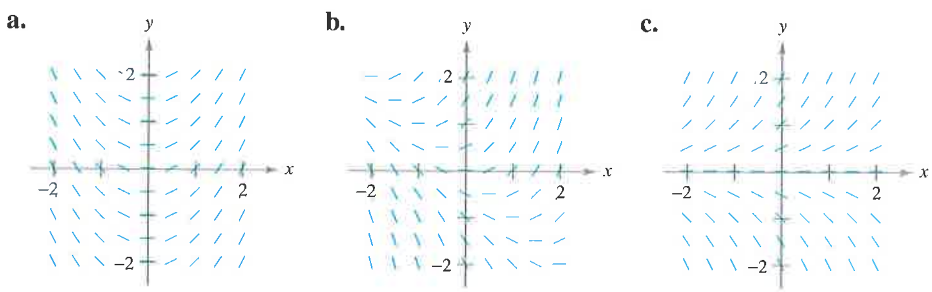


Figure 6.3

i. $y' = x + y$ **ii.** $y' = x$ **iii.** $y' = y$

Solution

a. From Figure 6.3(a), you can see that the slope at any point along the y-axis is 0. The only equation that satisfies this condition is $y' = x$. So, the graph matches (ii).

b. From Figure 6.3(b), you can see that the slope at the point $(1, -1)$ is 0. The only equation that satisfies this condition is $y' = x + y$. So, the graph matches (i).

c. From Figure 6.3(c), you can see that the slope at any point along the x-axis is 0. The only equation that satisfies this condition is $y' = y$. So, the graph matches (iii).

A solution curve of a differential equation $y' = F(x, y)$ is simply a curve in the xy-plane whose tangent line at each point (x, y) has slope equal to $F(x, y)$. This is illustrated in Example 5.

EXAMPLE 5 Sketching a Solution Using a Slope Field

Sketch a slope field for the differential equation

$$y' = 2x + y.$$

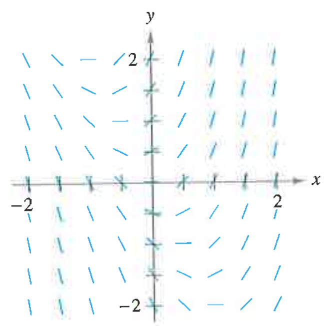
Use the slope field to sketch the solution that passes through the point $(1, 1)$.

Solution

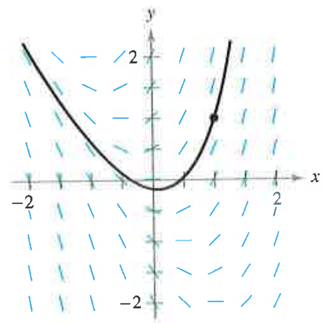
Make a table showing the slopes at several points. The table shown is a small sample. The slopes at many other points should be calculated to get a representative slope field.

x	-2	-2	-1	-1	0	0	1	1	2	2
y	-1	1	-1	1	-1	1	-1	1	-1	1
$y' = 2x + y$	-5	-3	-3	-1	-1	1	1	3	3	5

Next draw line segments at the points with their respective slopes, as shown in Figure 6.4.



Slope field for $y' = 2x + y$
Figure 6.4



Particular solution for $y' = 2x + y$ passing through $(1, 1)$
Figure 6.5

After the slope field is drawn, start at the initial point $(1, 1)$ and move to the right in the direction of the line segment. Continue to draw the solution curve so that it moves parallel to the nearby line segments. Do the same to the left of $(1, 1)$. The resulting solution is shown in Figure 6.5.

From Example 5, note that the slope field shows that y' increases to infinity as x increases.

NOTE Drawing a slope field by hand is tedious. In practice, slope fields are usually drawn using a graphing utility.

Euler's Method

Euler's Method is a numerical approach to approximating the particular solution of the differential equation

$$y' = F(x, y)$$

that passes through the point (x_0, y_0). From the given information, you know that the graph of the solution passes through the point (x_0, y_0) and has a slope of $F(x_0, y_0)$ at this point. This gives you a "starting point" for approximating the solution.

From this starting point, you can proceed in the direction indicated by the slope. Using a small step h, move along the tangent line until you arrive at the point (x_1, y_1), where

$$x_1 = x_0 + h \quad \text{and} \quad y_1 = y_0 + hF(x_0, y_0)$$

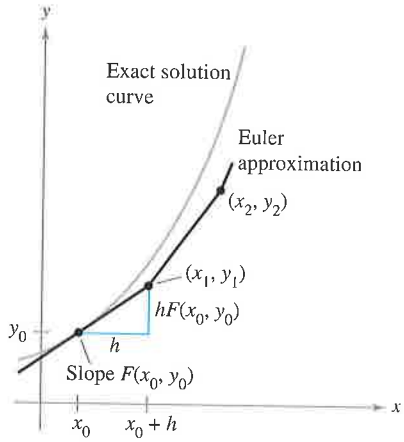


Figure 6.6

as shown in Figure 6.6. If you think of (x_1, y_1) as a new starting point, you can repeat the process to obtain a second point (x_2, y_2). The values of x_i and y_i are as follows.

$$\begin{array}{ll} x_1 = x_0 + h & y_1 = y_0 + hF(x_0, y_0) \\ x_2 = x_1 + h & y_2 = y_1 + hF(x_1, y_1) \\ \vdots & \vdots \\ x_n = x_{n-1} + h & y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \end{array}$$

NOTE You can obtain better approximations of the exact solution by choosing smaller and smaller step sizes.

EXAMPLE 6 Approximating a Solution Using Euler's Method

Use Euler's Method to approximate the particular solution of the differential equation

$$y' = x - y$$

passing through the point (0, 1). Use a step of $h = 0.1$.

Solution Using $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x - y$, you have $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3, \ldots,$ and

$$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.1)(0 - 1) = 0.9$$
$$y_2 = y_1 + hF(x_1, y_1) = 0.9 + (0.1)(0.1 - 0.9) = 0.82$$
$$y_3 = y_2 + hF(x_2, y_2) = 0.82 + (0.1)(0.2 - 0.82) = 0.758.$$

The first ten approximations are shown in the table. You can plot these values to see a graph of the approximate solution, as shown in Figure 6.7.

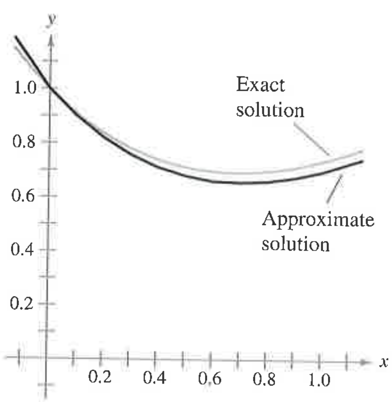


Figure 6.7

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	1	0.900	0.820	0.758	0.712	0.681	0.663	0.657	0.661	0.675	0.697

NOTE For the differential equation in Example 6, you can verify the exact solution to be $y = x - 1 + 2e^{-x}$. Figure 6.7 compares this exact solution with the approximate solution obtained in Example 6.

Exercises for Section 6.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, verify the solution of the differential equation.

Solution	*Differential Equation*
1. $y = Ce^{4x}$	$y' = 4y$
2. $y = e^{-x}$	$3y' + 4y = e^{-x}$
3. $x^2 + y^2 = Cy$	$y' = 2xy/(x^2 - y^2)$
4. $y^2 - 2\ln y = x^2$	$\dfrac{dy}{dx} = \dfrac{xy}{y^2 - 1}$
5. $y = C_1 \cos x + C_2 \sin x$	$y'' + y = 0$
6. $y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$	$y'' + 2y' + 2y = 0$
7. $y = -\cos x \ln\lvert\sec x + \tan x\rvert$	$y'' + y = \tan x$
8. $y = \frac{2}{5}(e^{-2x} + e^x)$	$y'' + 2y' = 2e^x$

In Exercises 9–12, verify the particular solution of the differential equation.

Solution	*Differential Equation and Initial Condition*
9. $y = \sin x \cos x - \cos^2 x$	$2y + y' = 2\sin(2x) - 1$, $\quad y\left(\dfrac{\pi}{4}\right) = 0$
10. $y = \frac{1}{2}x^2 - 4\cos x + 2$	$y' = x + 4\sin x$, $\quad y(0) = -2$
11. $y = 6e^{-2x^2}$	$y' = -4xy$, $\quad y(0) = 6$
12. $y = e^{-\cos x}$	$y' = y \sin x$, $\quad y\left(\dfrac{\pi}{2}\right) = 1$

In Exercises 13–18, determine whether the function is a solution of the differential equation $y^{(4)} - 16y = 0$.

13. $y = 3\cos x$

14. $y = 3\cos 2x$

15. $y = e^{-2x}$

16. $y = 5\ln x$

17. $y = C_1e^{2x} + C_2e^{-2x} + C_3 \sin 2x + C_4 \cos 2x$

18. $y = 3e^{2x} - 4\sin 2x$

In Exercises 19–24, determine whether the function is a solution of the differential equation $xy' - 2y = x^3e^x$.

19. $y = x^2$

20. $y = x^2e^x$

21. $y = x^2(2 + e^x)$

22. $y = \sin x$

23. $y = \ln x$

24. $y = x^2e^x - 5x^2$

In Exercises 25–28, some of the curves corresponding to different values of C in the general solution of the differential equation are given. Find the particular solution that passes through the point shown on the graph.

Solution	*Differential Equation*
25. $y = Ce^{-x/2}$	$2y' + y = 0$
26. $y(x^2 + y) = C$	$2xy + (x^2 + 2y)y' = 0$
27. $y^2 = Cx^3$	$2xy' - 3y = 0$
28. $2x^2 - y^2 = C$	$yy' - 2x = 0$

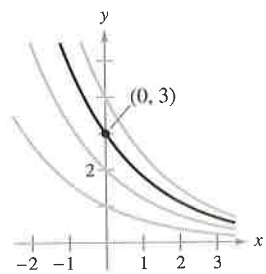


Figure for 25

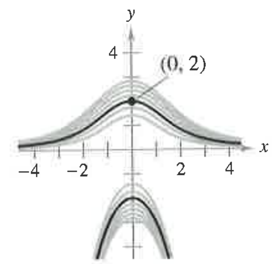


Figure for 26

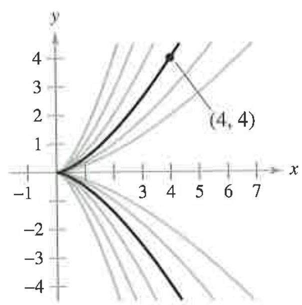


Figure for 27

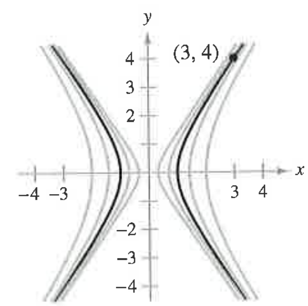


Figure for 28

In Exercises 29 and 30, the general solution of the differential equation is given. Use a graphing utility to graph the particular solutions for the given values of C.

29. $4yy' - x = 0$
$4y^2 - x^2 = C$
$C = 0, C = \pm 1, C = \pm 4$

30. $yy' + x = 0$
$x^2 + y^2 = C$
$C = 0, C = 1, C = 4$

In Exercises 31–36, verify that the general solution satisfies the differential equation. Then find the particular solution that satisfies the initial condition.

31. $y = Ce^{-2x}$
$y' + 2y = 0$
$y = 3$ when $x = 0$

32. $3x^2 + 2y^2 = C$
$3x + 2yy' = 0$
$y = 3$ when $x = 1$

33. $y = C_1 \sin 3x + C_2 \cos 3x$
$y'' + 9y = 0$
$y = 2$ when $x = \pi/6$
$y' = 1$ when $x = \pi/6$

34. $y = C_1 + C_2 \ln x$
$xy'' + y' = 0$
$y = 0$ when $x = 2$
$y' = \frac{1}{2}$ when $x = 2$

35. $y = C_1 x + C_2 x^3$
$x^2 y'' - 3xy' + 3y = 0$
$y = 0$ when $x = 2$
$y' = 4$ when $x = 2$

36. $y = e^{2x/3}(C_1 + C_2 x)$
$9y'' - 12y' + 4y = 0$
$y = 4$ when $x = 0$
$y = 0$ when $x = 3$

In Exercises 37–48, use integration to find a general solution of the differential equation.

37. $\dfrac{dy}{dx} = 3x^2$

38. $\dfrac{dy}{dx} = x^3 - 4x$

39. $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$

40. $\dfrac{dy}{dx} = \dfrac{e^x}{1 + e^x}$

41. $\dfrac{dy}{dx} = \dfrac{x - 2}{x}$

42. $\dfrac{dy}{dx} = x \cos x^2$

43. $\dfrac{dy}{dx} = \sin 2x$

44. $\dfrac{dy}{dx} = \tan^2 x$

45. $\dfrac{dy}{dx} = x\sqrt{x - 3}$

46. $\dfrac{dy}{dx} = x\sqrt{5 - x}$

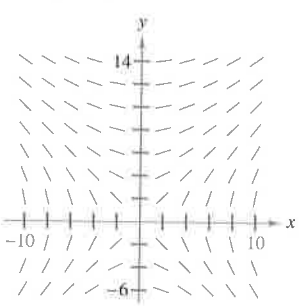
47. $\dfrac{dy}{dx} = xe^{x^2}$

48. $\dfrac{dy}{dx} = 5e^{-x/2}$

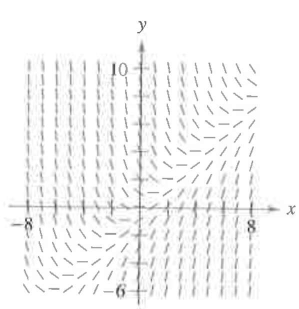
Slope Fields **In Exercises 49–52, a differential equation and its slope field are given. Determine the slopes (if possible) in the slope field at the points given in the table.**

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx						

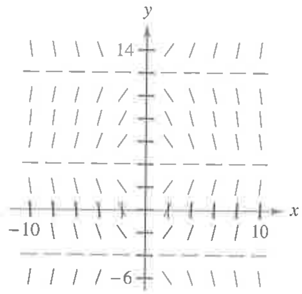
49. $\dfrac{dy}{dx} = \dfrac{x}{y}$



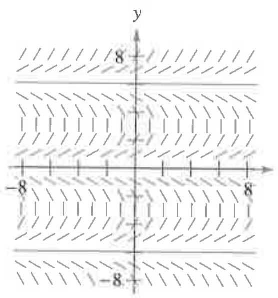
50. $\dfrac{dy}{dx} = x - y$



51. $\dfrac{dy}{dx} = x \cos \dfrac{\pi y}{8}$

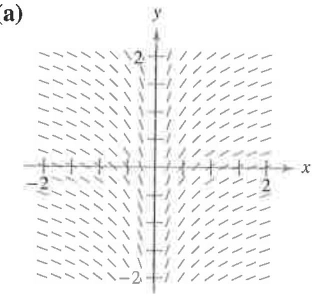


52. $\dfrac{dy}{dx} = \tan\left(\dfrac{\pi y}{6}\right)$

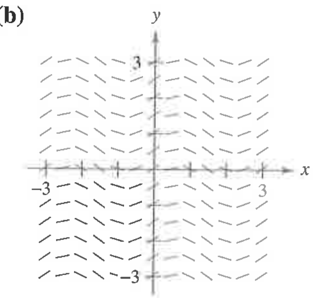


In Exercises 53–56, match the differential equation with its slope field. [The slope fields are labeled (a), (b), (c), and (d).]

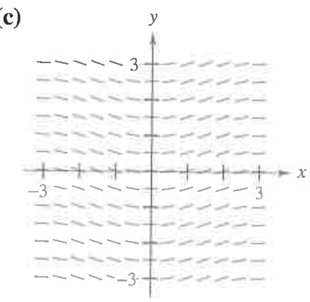
(a)



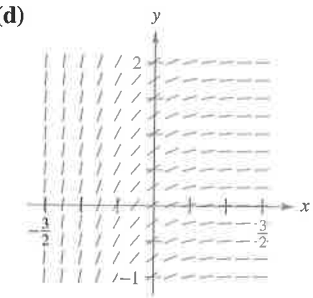
(b)



(c)



(d)



53. $\dfrac{dy}{dx} = \cos(2x)$

54. $\dfrac{dy}{dx} = \dfrac{1}{2}\sin x$

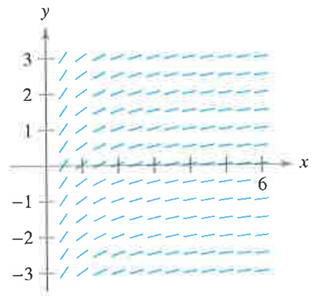
55. $\dfrac{dy}{dx} = e^{-2x}$

56. $\dfrac{dy}{dx} = \dfrac{1}{x}$

Slope Fields **In Exercises 57–60, (a) sketch the slope field for the differential equation, (b) use the slope field to sketch the solution that passes through the given point, and (c) discuss the graph of the solution as $x \to \infty$ and $x \to -\infty$.**

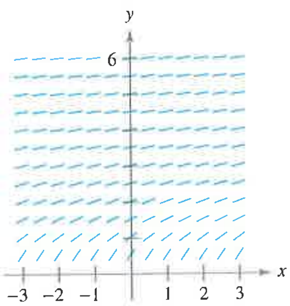
Differential Equation	*Point*
57. $y' = -x + 1$	$(2, 4)$
58. $y' = \frac{1}{3}x^2 - \frac{1}{2}x$	$(1, 1)$
59. $y' = y - 2x$	$(1, 1)$
60. $y' = y + xy$	$(0, 4)$

61. ***Slope Field*** Use the slope field for the differential equation $y' = 1/x$, where $x > 0$, to sketch the graph of the solution that satisfies each given initial condition. Then make a conjecture about the behavior of a particular solution of $y' = 1/x$ as $x \to \infty$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*



(a) $(1, 0)$ (b) $(2, -1)$

62. ***Slope Field*** Use the slope field for the differential equation $y' = 1/y$, where $y > 0$, to sketch the graph of the solution that satisfies each initial condition. Then make a conjecture about the behavior of a particular solution of $y' = 1/y$ as $x \to \infty$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com.*



(a) $(0, 1)$ (b) $(1, 1)$

Slope Fields **In Exercises 63–68, use a computer algebra system to (a) graph the slope field for the differential equation and (b) graph the solution satisfying the specified initial condition.**

63. $\dfrac{dy}{dx} = 0.5y, \quad y(0) = 6$

64. $\dfrac{dy}{dx} = 2 - y, \quad y(0) = 4$

65. $\dfrac{dy}{dx} = 0.02y(10 - y), \quad y(0) = 2$

66. $\dfrac{dy}{dx} = 0.2x(2 - y), \quad y(0) = 9$

67. $\dfrac{dy}{dx} = 0.4y(3 - x), \quad y(0) = 1$

68. $\dfrac{dy}{dx} = \dfrac{1}{2}e^{-x/8} \sin \dfrac{\pi y}{4}, \quad y(0) = 2$

Euler's Method **In Exercises 69–74, use Euler's Method to make a table of values for the approximate solution of the differential equation with the specified initial value. Use n steps of size h.**

69. $y' = x + y, \quad y(0) = 2, \quad n = 10, \quad h = 0.1$

70. $y' = x + y, \quad y(0) = 2, \quad n = 20, \quad h = 0.05$

71. $y' = 3x - 2y, \quad y(0) = 3, \quad n = 10, \quad h = 0.05$

72. $y' = 0.5x(3 - y), \quad y(0) = 1, \quad n = 5, \quad h = 0.4$

73. $y' = e^{xy}, \quad y(0) = 1, \quad n = 10, \quad h = 0.1$

74. $y' = \cos x + \sin y, \quad y(0) = 5, \quad n = 10, \quad h = 0.1$

In Exercises 75–77, complete the table using the exact solution of the differential equation and two approximations obtained using Euler's Method to approximate the particular solution of the differential equation. Use $h = 0.2$ and 0.1 and compute each approximation to four decimal places.

x	0	0.2	0.4	0.6	0.8	1.0
$y(x)$ (exact)						
$y(x)$ ($h = 0.2$)						
$y(x)$ ($h = 0.1$)						

	Differential Equation	*Initial Condition*	*Exact Solution*
75.	$\dfrac{dy}{dx} = y$	$(0, 3)$	$y = 3e^x$
76.	$\dfrac{dy}{dx} = \dfrac{2x}{y}$	$(0, 2)$	$y = \sqrt{2x^2 + 4}$
77.	$\dfrac{dy}{dx} = y + \cos(x)$	$(0, 0)$	$y = \frac{1}{2}(\sin x - \cos x + e^x)$

78. Compare the values of the approximations in Exercises 75–77 with the values given by the exact solution. How does the error change as h increases?

79. ***Temperature*** At time $t = 0$ minutes, the temperature of an object is 140°F. The temperature of the object is changing at the rate given by the differential equation

$$\frac{dy}{dt} = -\frac{1}{2}(y - 72).$$

(a) Use a graphing utility and Euler's Method to approximate the particular solutions of this differential equation at $t = 1$, 2, and 3. Use a step size of $h = 0.1$. (A graphing utility program for Euler's Method is available on the website *college.hmco.com.*)

(b) Compare your results with the exact solution

$$y = 72 + 68e^{-t/2}.$$

80. ***Temperature*** Repeat Exercise 79 using a step size of $h = 0.05$. Compare the results.

Writing About Concepts

81. In your own words, describe the difference between a general solution of a differential equation and a particular solution.

82. Explain how to interpret a slope field.

83. Describe how to use Euler's Method to approximate the particular solution of a differential equation.

84. It is known that $y = Ce^{kx}$ is a solution of the differential equation $y' = 0.07y$. Is it possible to determine C or k from the information given? If so, find its value.

True or False? **In Exercises 85–88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

85. If $y = f(x)$ is a solution of a first-order differential equation, then $y = f(x) + C$ is also a solution.

86. The general solution of a differential equation is $y = -4.9x^2 + C_1x + C_2$. To find a particular solution, you must be given two initial conditions.

87. Slope fields represent the general solutions of differential equations.

88. A slope field shows that the slope at the point $(1, 1)$ is 6. This slope field represents the family of solutions for the differential equation $y' = 4x + 2y$.

89. ***Error and Euler's Method*** The exact solution of the differential equation

$$\frac{dy}{dx} = -2y$$

where $y(0) = 4$, is $y = 4e^{-2x}$.

(a) Use a graphing utility to complete the table, where y is the exact value of the solution, y_1 is the approximate solution using Euler's Method with $h = 0.1$, y_2 is the approximate solution using Euler's Method with $h = 0.2$, e_1 is the absolute error $|y - y_1|$, e_2 is the absolute error $|y - y_2|$, and r is the ratio e_1/e_2.

x	0	0.2	0.4	0.6	0.8	1
y						
y_1						
y_2						
e_1						
e_2						
r						

(b) What can you conclude about the ratio r as h changes?

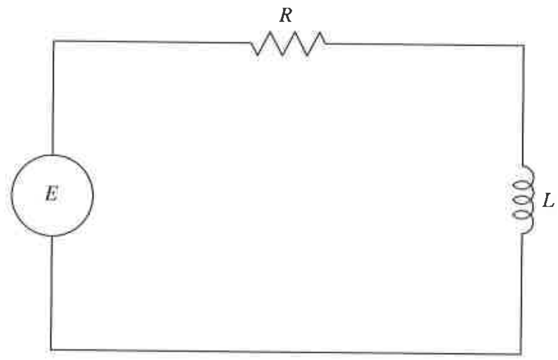
(c) Predict the absolute error when $h = 0.05$.

90. ***Error and Euler's Method*** Repeat Exercise 89 where the exact solution of the differential equation

$$\frac{dy}{dx} = x - y$$

where $y(0) = 1$, is $y = x - 1 + 2e^{-x}$.

91. ***Electric Circuits*** The diagram shows a simple electric circuit consisting of a power source, a resistor, and an inductor.



A model of the current I, in amperes (A), at time t is given by the first-order differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

where $E(t)$ is the voltage (V) produced by the power source, R is the resistance, in ohms (Ω), and L is the inductance, in henrys (H). Suppose the electric circuit consists of a 24-V power source, a 12-Ω resistor, and a 4-H inductor.

(a) Sketch a slope field for the differential equation.

(b) What is the limiting value of the current? Explain.

92. ***Think About It*** It is known that $y = e^{kt}$ is a solution of the differential equation $y'' - 16y = 0$. Find the values of k.

93. ***Think About It*** It is known that $y = A \sin \omega t$ is a solution of the differential equation $y'' + 16y = 0$. Find the values of ω.

Putnam Exam Challenge

94. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x)$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

95. Prove that if the family of integral curves of the differential equation

$$\frac{dy}{dx} + p(x)y = q(x), \qquad p(x) \cdot q(x) \neq 0$$

is cut by the line $x = k$, the tangents at the points of intersection are concurrent.

Section 6.2

Differential Equations: Growth and Decay

- **Use separation of variables to solve a simple differential equation.**
- **Use exponential functions to model growth and decay in applied problems.**

Differential Equations

In the preceding section, you learned to analyze visually the solutions of differential equations using slope fields and to approximate solutions numerically using Euler's Method. Analytically, you have learned to solve only two types of differential equations—those of the forms

$$y' = f(x) \quad \text{and} \quad y'' = f(x).$$

In this section, you will learn how to solve a more general type of differential equation. The strategy is to rewrite the equation so that each variable occurs on only one side of the equation. This strategy is called *separation of variables*. (You will study this strategy in detail in Section 6.3.)

NOTE When you integrate both sides of the equation in Example 1, you don't need to add a constant of integration to both sides of the equation. If you did, you would obtain the same result as in Example 1.

$$\int y\,dy = \int 2x\,dx$$
$$\tfrac{1}{2}y^2 + C_2 = x^2 + C_3$$
$$\tfrac{1}{2}y^2 = x^2 + (C_3 - C_2)$$
$$\tfrac{1}{2}y^2 = x^2 + C_1$$

EXAMPLE 1 Solving a Differential Equation

Solve the differential equation $y' = 2x/y$.

Solution

$$y' = \frac{2x}{y} \qquad \text{Write original equation.}$$
$$yy' = 2x \qquad \text{Multiply both sides by } y.$$
$$\int yy'\,dx = \int 2x\,dx \qquad \text{Integrate with respect to } x.$$
$$\int y\,dy = \int 2x\,dx \qquad dy = y'\,dx$$
$$\frac{1}{2}y^2 = x^2 + C_1 \qquad \text{Apply Power Rule.}$$
$$y^2 - 2x^2 = C \qquad \text{Rewrite, letting } C = 2C_1.$$

So, the general solution is given by

$$y^2 - 2x^2 = C.$$

You can use implicit differentiation to check this result.

EXPLORATION

In Example 1, the general solution of the differential equation is

$$y^2 - 2x^2 = C.$$

Use a graphing utility to sketch several particular solutions—those given by $C = \pm 2$, $C = \pm 1$, and $C = 0$. Describe the solutions graphically. Is the following statement true of each solution?

The slope of the graph at the point (x, y) is equal to twice the ratio of x and y.

Explain your reasoning. Are all curves for which this statement is true represented by the general solution?

In practice, most people prefer to use Leibniz notation and differentials when applying separation of variables. The solution of Example 1 is shown below using this notation.

$$\frac{dy}{dx} = \frac{2x}{y}$$
$$y\,dy = 2x\,dx$$
$$\int y\,dy = \int 2x\,dx$$
$$\frac{1}{2}y^2 = x^2 + C_1$$
$$y^2 - 2x^2 = C$$

Growth and Decay Models

In many applications, the rate of change of a variable y is proportional to the value of y. If y is a function of time t, the proportion can be written as shown.

$$\underbrace{\frac{dy}{dt}}_{\text{Rate of change of } y} \underbrace{=}_{\text{is}} \underbrace{ky}_{\text{proportional to } y.}$$

The general solution of this differential equation is given in the following theorem.

> **THEOREM 6.1 Exponential Growth and Decay Model**
>
> If y is a differentiable function of t such that $y > 0$ and $y' = ky$, for some constant k, then
>
> $$y = Ce^{kt}.$$
>
> C is the **initial value** of y, and k is the **proportionality constant. Exponential growth** occurs when $k > 0$, and **exponential decay** occurs when $k < 0$.

NOTE Differentiate the function $y = Ce^{kt}$ with respect to t, and verify that $y' = ky$.

Proof

$$y' = ky \qquad \text{Write original equation.}$$

$$\frac{y'}{y} = k \qquad \text{Separate variables.}$$

$$\int \frac{y'}{y}\,dt = \int k\,dt \qquad \text{Integrate with respect to } t.$$

$$\int \frac{1}{y}\,dy = \int k\,dt \qquad dy = y'\,dt$$

$$\ln y = kt + C_1 \qquad \text{Find antiderivative of each side.}$$

$$y = e^{kt}e^{C_1} \qquad \text{Solve for } y.$$

$$y = Ce^{kt} \qquad \text{Let } C = e^{C_1}.$$

So, all solutions of $y' = ky$ are of the form $y = Ce^{kt}$.

EXAMPLE 2 Using an Exponential Growth Model

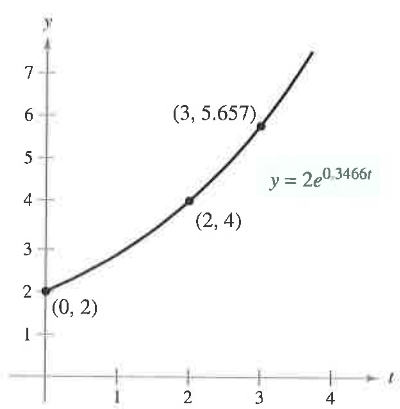
The rate of change of y is proportional to y. When $t = 0$, $y = 2$. When $t = 2$, $y = 4$. What is the value of y when $t = 3$?

Solution Because $y' = ky$, you know that y and t are related by the equation $y = Ce^{kt}$. You can find the values of the constants C and k by applying the initial conditions.

$$2 = Ce^0 \quad \Rightarrow \quad C = 2 \qquad \text{When } t = 0,\ y = 2.$$

$$4 = 2e^{2k} \quad \Rightarrow \quad k = \frac{1}{2}\ln 2 \approx 0.3466 \qquad \text{When } t = 2,\ y = 4.$$

So, the model is $y \approx 2e^{0.3466t}$. When $t = 3$, the value of y is $2e^{0.3466(3)} \approx 5.657$ (see Figure 6.8).



If the rate of change of y is proportional to y, then y follows an exponential model.
Figure 6.8

STUDY TIP Using logarithmic properties, note that the value of k in Example 2 can also be written as $\ln(\sqrt{2})$. So, the model becomes $y = 2e^{(\ln\sqrt{2})t}$, which can then be rewritten as $y = 2(\sqrt{2})^t$.

TECHNOLOGY Most graphing utilities have curve-fitting capabilities that can be used to find models that represent data. Use the *exponential regression* feature of a graphing utility and the information in Example 2 to find a model for the data. How does your model compare with the given model?

Radioactive decay is measured in terms of *half-life*—the number of years required for half of the atoms in a sample of radioactive material to decay. The half-lives of some common radioactive isotopes are shown below.

Uranium (^{238}U)	4,470,000,000 years
Plutonium (^{239}Pu)	24,100 years
Carbon (^{14}C)	5715 years
Radium (^{226}Ra)	1599 years
Einsteinium (^{254}Es)	276 days
Nobelium (^{257}No)	25 seconds

EXAMPLE 3 Radioactive Decay


Sergei Supinsky/AFP/Getty Images

Suppose that 10 grams of the plutonium isotope Pu-239 was released in the Chernobyl nuclear accident. How long will it take for the 10 grams to decay to 1 gram?

Solution Let y represent the mass (in grams) of the plutonium. Because the rate of decay is proportional to y, you know that

$$y = Ce^{kt}$$

where t is the time in years. To find the values of the constants C and k, apply the initial conditions. Using the fact that $y = 10$ when $t = 0$, you can write

$$10 = Ce^{k(0)} = Ce^0$$

which implies that $C = 10$. Next, using the fact that $y = 5$ when $t = 24{,}100$, you can write

$$\begin{aligned} 5 &= 10e^{k(24,100)} \\ \frac{1}{2} &= e^{24,100k} \\ \frac{1}{24,100}\ln\frac{1}{2} &= k \\ -0.000028761 &\approx k. \end{aligned}$$

So, the model is

$$y = 10e^{-0.000028761t}. \qquad \text{Half-life model}$$

To find the time it would take for 10 grams to decay to 1 gram, you can solve for t in the equation

$$1 = 10e^{-0.000028761t}.$$

The solution is approximately 80,059 years.

NOTE The exponential decay model in Example 3 could also be written as $y = 10\left(\frac{1}{2}\right)^{t/24,100}$. This model is much easier to derive, but for some applications it is not as convenient to use.

From Example 3, notice that in an exponential growth or decay problem, it is easy to solve for C when you are given the value of y at $t = 0$. The next example demonstrates a procedure for solving for C and k when you do not know the value of y at $t = 0$.

EXAMPLE 4 Population Growth

Suppose an experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

Solution Let $y = Ce^{kt}$ be the number of flies at time t, where t is measured in days. Because $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$, you can write

$$100 = Ce^{2k} \quad \text{and} \quad 300 = Ce^{4k}.$$

From the first equation, you know that $C = 100e^{-2k}$. Substituting this value into the second equation produces the following.

$$\begin{aligned} 300 &= 100e^{-2k}e^{4k} \\ 300 &= 100e^{2k} \\ \ln 3 &= 2k \\ \frac{1}{2}\ln 3 &= k \\ 0.5493 &\approx k \end{aligned}$$

So, the exponential growth model is

$$y = Ce^{0.5493t}.$$

To solve for C, reapply the condition $y = 100$ when $t = 2$ and obtain

$$\begin{aligned} 100 &= Ce^{0.5493(2)} \\ C &= 100e^{-1.0986} \approx 33. \end{aligned}$$

So, the original population (when $t = 0$) consisted of approximately $y = C = 33$ flies, as shown in Figure 6.9.

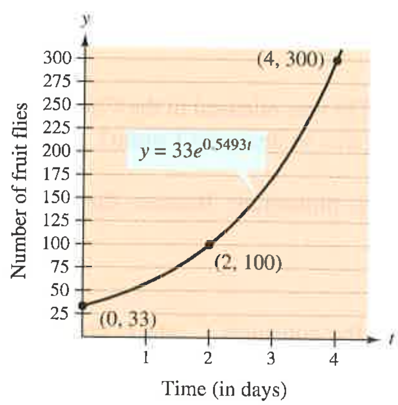


Figure 6.9

EXAMPLE 5 Declining Sales

Four months after it stops advertising, a manufacturing company notices that its sales have dropped from 100,000 units per month to 80,000 units per month. If the sales follow an exponential pattern of decline, what will they be after another 2 months?

Solution Use the exponential decay model $y = Ce^{kt}$, where t is measured in months. From the initial condition ($t = 0$), you know that $C = 100{,}000$. Moreover, because $y = 80{,}000$ when $t = 4$, you have

$$\begin{aligned} 80{,}000 &= 100{,}000e^{4k} \\ 0.8 &= e^{4k} \\ \ln(0.8) &= 4k \\ -0.0558 &\approx k. \end{aligned}$$

So, after 2 more months ($t = 6$), you can expect the monthly sales rate to be

$$\begin{aligned} y &\approx 100{,}000e^{-0.0558(6)} \\ &\approx 71{,}500 \text{ units.} \end{aligned}$$

See Figure 6.10.

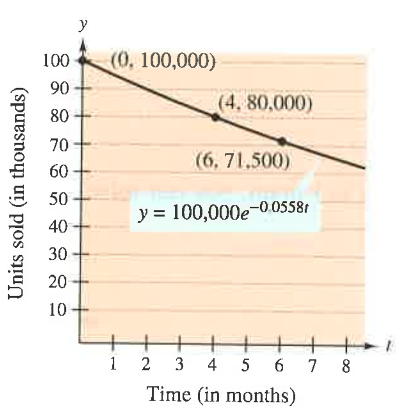


Figure 6.10

In Examples 2 through 5, you did not actually have to solve the differential equation

$$y' = ky.$$

(This was done once in the proof of Theorem 6.1.) The next example demonstrates a problem whose solution involves the separation of variables technique. The example concerns **Newton's Law of Cooling,** which states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium.

EXAMPLE 6 Newton's Law of Cooling

Let y represent the temperature (in °F) of an object in a room whose temperature is kept at a constant 60°. If the object cools from 100° to 90° in 10 minutes, how much longer will it take for its temperature to decrease to 80°?

Solution From Newton's Law of Cooling, you know that the rate of change in y is proportional to the difference between y and 60. This can be written as

$$y' = k(y - 60), \qquad 80 \le y \le 100.$$

To solve this differential equation, use separation of variables, as shown.

$$\frac{dy}{dt} = k(y - 60) \qquad \text{Differential equation}$$

$$\left(\frac{1}{y - 60}\right) dy = k\,dt \qquad \text{Separate variables.}$$

$$\int \frac{1}{y - 60}\,dy = \int k\,dt \qquad \text{Integrate each side.}$$

$$\ln|y - 60| = kt + C_1 \qquad \text{Find antiderivative of each side.}$$

Because $y > 60$, $|y - 60| = y - 60$, and you can omit the absolute value signs. Using exponential notation, you have

$$y - 60 = e^{kt + C_1} \quad \Longrightarrow \quad y = 60 + Ce^{kt}. \qquad C = e^{C_1}$$

Using $y = 100$ when $t = 0$, you obtain $100 = 60 + Ce^{k(0)} = 60 + C$, which implies that $C = 40$. Because $y = 90$ when $t = 10$,

$$90 = 60 + 40e^{k(10)}$$

$$30 = 40e^{10k}$$

$$k = \tfrac{1}{10}\ln\tfrac{3}{4} \approx -0.02877.$$

So, the model is

$$y = 60 + 40e^{-0.02877t} \qquad \text{Cooling model}$$

and finally, when $y = 80$, you obtain

$$80 = 60 + 40e^{-0.02877t}$$

$$20 = 40e^{-0.02877t}$$

$$\tfrac{1}{2} = e^{-0.02877t}$$

$$\ln\tfrac{1}{2} = -0.02877t$$

$$t \approx 24.09 \text{ minutes.}$$

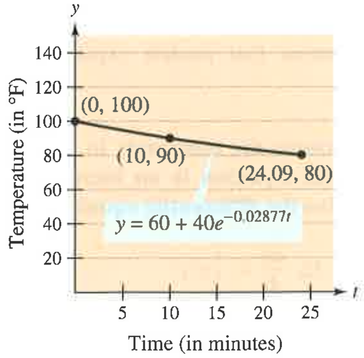


Figure 6.11

So, it will require about 14.09 *more* minutes for the object to cool to a temperature of 80° (see Figure 6.11).

Exercises for Section 6.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–10, solve the differential equation.

1. $\frac{dy}{dx} = x + 2$

2. $\frac{dy}{dx} = 4 - x$

3. $\frac{dy}{dx} = y + 2$

4. $\frac{dy}{dx} = 4 - y$

5. $y' = \frac{5x}{y}$

6. $y' = \frac{\sqrt{x}}{3y}$

7. $y' = \sqrt{x}\,y$

8. $y' = x(1 + y)$

9. $(1 + x^2)y' - 2xy = 0$

10. $xy + y' = 100x$

In Exercises 11–14, write and solve the differential equation that models the verbal statement.

11. The rate of change of Q with respect to t is inversely proportional to the square of t.

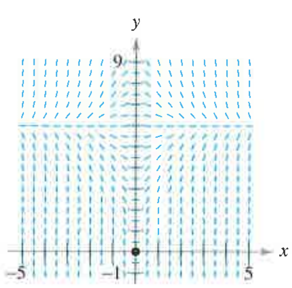
12. The rate of change of P with respect to t is proportional to $10 - t$.

13. The rate of change of N with respect to s is proportional to $250 - s$.

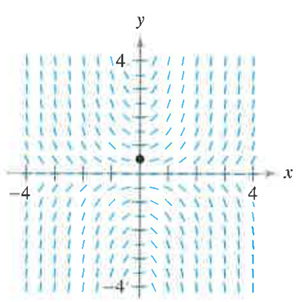
14. The rate of change of y with respect to x varies jointly as x and $L - y$.

Slope Fields **In Exercises 15 and 16, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

15. $\frac{dy}{dx} = x(6 - y)$, $(0, 0)$



16. $\frac{dy}{dx} = xy$, $\left(0, \frac{1}{2}\right)$



In Exercises 17–20, find the function $y = f(t)$ passing through the point (0, 10) with the given first derivative. Use a graphing utility to graph the solution.

17. $\frac{dy}{dt} = \frac{1}{2}t$

18. $\frac{dy}{dt} = -\frac{3}{4}\sqrt{t}$

19. $\frac{dy}{dt} = -\frac{1}{2}y$

20. $\frac{dy}{dt} = \frac{3}{4}y$

In Exercises 21–24, write and solve the differential equation that models the verbal statement. Evaluate the solution at the specified value of the independent variable.

21. The rate of change of y is proportional to y. When $x = 0$, $y = 4$ and when $x = 3$, $y = 10$. What is the value of y when $x = 6$?

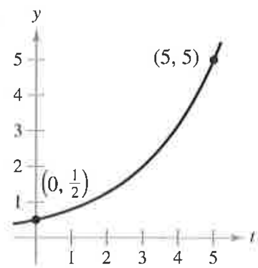
22. The rate of change of N is proportional to N. When $t = 0$, $N = 250$ and when $t = 1$, $N = 400$. What is the value of N when $t = 4$?

23. The rate of change of V is proportional to V. When $t = 0$, $V = 20{,}000$ and when $t = 4$, $V = 12{,}500$. What is the value of V when $t = 6$?

24. The rate of change of P is proportional to P. When $t = 0$, $P = 5000$ and when $t = 1$, $P = 4750$. What is the value of P when $t = 5$?

In Exercises 25–28, find the exponential function $y = Ce^{kt}$ that passes through the two given points.

25.



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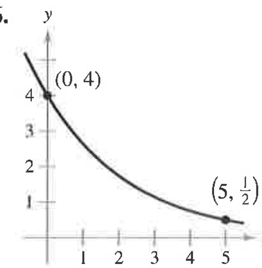

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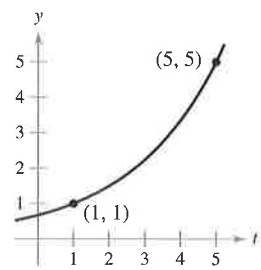

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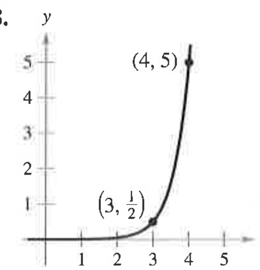


Writing About Concepts

29. Describe what the values of C and k represent in the exponential growth and decay model, $y = Ce^{kt}$.

30. Give the differential equation that models exponential growth and decay.

In Exercises 31 and 32, determine the quadrants in which the solution of the differential equation is an increasing function. Explain. (Do not solve the differential equation.)

31. $\frac{dy}{dx} = \frac{1}{2}xy$

32. $\frac{dy}{dx} = \frac{1}{2}x^2y$

Radioactive Decay **In Exercises 33–40, complete the table for the radioactive isotope.**

	Isotope	*Half-Life (in years)*	*Initial Quantity*	*Amount After 1000 Years*	*Amount After 10,000 Years*
33.	^{226}Ra	1599	10 g		
34.	^{226}Ra	1599		1.5 g	
35.	^{226}Ra	1599			0.5 g
36.	^{14}C	5715			2 g
37.	^{14}C	5715	5 g		
38.	^{14}C	5715		3.2 g	
39.	^{239}Pu	24,100		2.1 g	
40.	^{239}Pu	24,100			0.4 g

41. ***Radioactive Decay*** Radioactive radium has a half-life of approximately 1599 years. What percent of a given amount remains after 100 years?

42. ***Carbon Dating*** Carbon-14 dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, the amount of ^{14}C absorbed by a tree that grew several centuries ago should be the same as the amount of ^{14}C absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much of the radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal? (The half-life of ^{14}C is 5715 years.)

Compound Interest **In Exercises 43–48, complete the table for a savings account in which interest is compounded continuously.**

	Initial Investment	*Annual Rate*	*Time to Double*	*Amount After 10 Years*
43.	\$1000	6%		
44.	\$20,000	$5\frac{1}{2}\%$		
45.	\$750		$7\frac{3}{4}$ yr	
46.	\$10,000		5 yr	
47.	\$500			\$1292.85
48.	\$2000			\$5436.56

Compound Interest **In Exercises 49–52, find the principal P that must be invested at rate r, compounded monthly, so that \$500,000 will be available for retirement in t years.**

49. $r = 7\frac{1}{2}\%,\quad t = 20$

50. $r = 6\%,\quad t = 40$

51. $r = 8\%,\quad t = 35$

52. $r = 9\%,\quad t = 25$

Compound Interest **In Exercises 53–56, find the time necessary for \$1000 to double if it is invested at a rate of r compounded (a) annually, (b) monthly, (c) daily, and (d) continuously.**

53. $r = 7\%$

54. $r = 6\%$

55. $r = 8.5\%$

56. $r = 5.5\%$

Population **In Exercises 57–60, the population (in millions) of a country in 2001 and the expected continuous annual rate of change k of the population for the years 2000 through 2010 are given.** ***(Source: U.S. Census Bureau, International Data Base)***

(a) Find the exponential growth model $P = Ce^{kt}$ for the population by letting $t = 0$ correspond to 2000.

(b) Use the model to predict the population of the country in 2015.

(c) Discuss the relationship between the sign of k and the change in population for the country.

	Country	*2001 Population*	k
57.	Bulgaria	7.7	−0.009
58.	Cambodia	12.7	0.018
59.	Jordan	5.2	0.026
60.	Lithuania	3.6	−0.002

61. ***Modeling Data*** One hundred bacteria are started in a culture and the number N of bacteria is counted each hour for 5 hours. The results are shown in the table, where t is the time in hours.

t	0	1	2	3	4	5
N	100	126	151	198	243	297

(a) Use the regression capabilities of a graphing utility to find an exponential model for the data.

(b) Use the model to estimate the time required for the population to quadruple in size.

62. ***Bacteria Growth*** The number of bacteria in a culture is increasing according to the law of exponential growth. There are 125 bacteria in the culture after 2 hours and 350 bacteria after 4 hours.

(a) Find the initial population.

(b) Write an exponential growth model for the bacteria population. Let t represent time in hours.

(c) Use the model to determine the number of bacteria after 8 hours.

(d) After how many hours will the bacteria count be 25,000?

63. ***Learning Curve*** The management at a certain factory has found that a worker can produce at most 30 units in a day. The learning curve for the number of units N produced per day after a new employee has worked t days is $N = 30(1 - e^{kt})$. After 20 days on the job, a particular worker produces 19 units.

(a) Find the learning curve for this worker.

(b) How many days should pass before this worker is producing 25 units per day?

64. ***Learning Curve*** If in Exercise 63 management requires a new employee to produce at least 20 units per day after 30 days on the job, find (a) the learning curve that describes this minimum requirement and (b) the number of days before a minimal achiever is producing 25 units per day.

65. ***Modeling Data*** The table shows the population P (in millions) of the United States from 1960 to 2000. *(Source: U.S. Census Bureau)*

Year	1960	1970	1980	1990	2000
Population, P	181	205	228	250	282

(a) Use the 1960 and 1970 data to find an exponential model P_1 for the data. Let $t = 0$ represent 1960.

(b) Use a graphing utility to find an exponential model P_2 for the data. Let $t = 0$ represent 1960.

(c) Use a graphing utility to plot the data and graph both models in the same viewing window. Compare the actual data with the predictions. Which model better fits the data?

(d) Estimate when the population will be 320 million.

66. ***Modeling Data*** The table shows the net receipts and the amounts required to service the national debt (interest on Treasury debt securities) of the United States from 1992 through 2001. The monetary amounts are given in billions of dollars. *(Source: U.S. Office of Management and Budget)*

Year	1992	1993	1994	1995	1996
Receipts	1091.3	1154.4	1258.6	1351.8	1453.1
Interest	292.3	292.5	296.3	332.4	343.9

Year	1997	1998	1999	2000	2001
Receipts	1579.3	1721.8	1827.5	2025.2	1991.2
Interest	355.8	363.8	353.5	361.9	359.5

(a) Use the regression capabilities of a graphing utility to find an exponential model R for the receipts and a quartic model I for the amount required to service the debt. Let t represent the time in years, with $t = 2$ corresponding to 1992.

(b) Use a graphing utility to plot the points corresponding to the receipts, and graph the corresponding model. Based on the model, what is the continuous rate of growth of the receipts?

(c) Use a graphing utility to plot the points corresponding to the amount required to service the debt, and graph the quartic model.

(d) Find a function $P(t)$ that approximates the percent of the receipts that is required to service the national debt. Use a graphing utility to graph this function.

67. ***Sound Intensity*** The level of sound β (in decibels), with an intensity of I is

$$\beta(I) = 10 \log_{10} \frac{I}{I_0}$$

where I_0 is an intensity of 10^{-16} watts per square centimeter, corresponding roughly to the faintest sound that can be heard. Determine $\beta(I)$ for the following.

(a) $I = 10^{-14}$ watts per square centimeter (whisper)

(b) $I = 10^{-9}$ watts per square centimeter (busy street corner)

(c) $I = 10^{-6.5}$ watts per square centimeter (air hammer)

(d) $I = 10^{-4}$ watts per square centimeter (threshold of pain)

68. ***Noise Level*** With the installation of noise suppression materials, the noise level in an auditorium was reduced from 93 to 80 decibels. Use the function in Exercise 67 to find the percent decrease in the intensity level of the noise as a result of the installation of these materials.

69. ***Forestry*** The value of a tract of timber is

$$V(t) = 100{,}000e^{0.8\sqrt{t}}$$

where t is the time in years, with $t = 0$ corresponding to 1998. If money earns interest continuously at 10%, the present value of the timber at any time t is $A(t) = V(t)e^{-0.10t}$. Find the year in which the timber should be harvested to maximize the present value function.

70. ***Earthquake Intensity*** On the Richter scale, the magnitude R of an earthquake of intensity I is

$$R = \frac{\ln I - \ln I_0}{\ln 10}$$

where I_0 is the minimum intensity used for comparison. Assume that $I_0 = 1$.

(a) Find the intensity of the 1906 San Francisco earthquake ($R = 8.3$).

(b) Find the factor by which the intensity is increased if the Richter scale measurement is doubled.

(c) Find dR/dI.

71. ***Newton's Law of Cooling*** When an object is removed from a furnace and placed in an environment with a constant temperature of 80°F, its core temperature is 1500°F. One hour after it is removed, the core temperature is 1120°F. Find the core temperature 5 hours after the object is removed from the furnace.

72. ***Newton's Law of Cooling*** A container of hot liquid is placed in a freezer that is kept at a constant temperature of 20°F. The initial temperature of the liquid is 160°F. After 5 minutes, the liquid's temperature is 60°F. How much longer will it take for its temperature to decrease to 30°F?

True or False? **In Exercises 73–76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

73. In exponential growth, the rate of growth is constant.

74. In linear growth, the rate of growth is constant.

75. If prices are rising at a rate of 0.5% per month, then they are rising at a rate of 6% per year.

76. The differential equation modeling exponential growth is $dy/dx = ky$, where k is a constant.

Section 6.3 Separation of Variables and the Logistic Equation

- **Recognize and solve differential equations that can be solved by separation of variables.**
- **Recognize and solve homogeneous differential equations.**
- **Use differential equations to model and solve applied problems.**
- **Solve and analyze logistic differential equations.**

Separation of Variables

Consider a differential equation that can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

where M is a continuous function of x alone and N is a continuous function of y alone. As you saw in the preceding section, for this type of equation, all x terms can be collected with dx and all y terms with dy, and a solution can be obtained by integration. Such equations are said to be **separable,** and the solution procedure is called *separation of variables*. Below are some examples of differential equations that are separable.

Original Differential Equation	*Rewritten with Variables Separated*
$x^2 + 3y\dfrac{dy}{dx} = 0$	$3y\,dy = -x^2\,dx$
$(\sin x)y' = \cos x$	$dy = \cot x\,dx$
$\dfrac{xy'}{e^y + 1} = 2$	$\dfrac{1}{e^y + 1}\,dy = \dfrac{2}{x}\,dx$

EXAMPLE 1 Separation of Variables

Find the general solution of $(x^2 + 4)\dfrac{dy}{dx} = xy$.

Solution To begin, note that $y = 0$ is a solution. To find other solutions, assume that $y \neq 0$ and separate variables as shown.

$$(x^2 + 4)\,dy = xy\,dx \qquad \text{Differential form}$$

$$\frac{dy}{y} = \frac{x}{x^2 + 4}\,dx \qquad \text{Separate variables.}$$

Now, integrate to obtain

$$\int \frac{dy}{y} = \int \frac{x}{x^2 + 4}\,dx \qquad \text{Integrate.}$$

$$\ln|y| = \frac{1}{2}\ln(x^2 + 4) + C_1$$

$$\ln|y| = \ln\sqrt{x^2 + 4} + C_1$$

$$|y| = e^{C_1}\sqrt{x^2 + 4}$$

$$y = \pm e^{C_1}\sqrt{x^2 + 4}.$$

Because $y = 0$ is also a solution, you can write the general solution as

$$y = C\sqrt{x^2 + 4}. \qquad \text{General solution}$$

NOTE Be sure to check your solutions throughout this chapter. In Example 1, you can check the solution $y = C\sqrt{x^2 + 4}$ by differentiating and substituting into the original equation.

$$(x^2 + 4)\frac{dy}{dx} = xy$$

$$(x^2 + 4)\frac{Cx}{\sqrt{x^2 + 4}} \stackrel{?}{=} x\left(C\sqrt{x^2 + 4}\right)$$

$$Cx\sqrt{x^2 + 4} = Cx\sqrt{x^2 + 4}$$

So, the solution checks.

In some cases it is not feasible to write the general solution in the explicit form $y = f(x)$. The next example illustrates such a solution. Implicit differentiation can be used to verify this solution.

FOR FURTHER INFORMATION For an example (from engineering) of a differential equation that is separable, see the article "Designing a Rose Cutter" by J. S. Hartzler in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

EXAMPLE 2 Finding a Particular Solution

Given the initial condition $y(0) = 1$, find the particular solution of the equation

$$xy\,dx + e^{-x^2}(y^2 - 1)\,dy = 0.$$

Solution Note that $y = 0$ is a solution of the differential equation—but this solution does not satisfy the initial condition. So, you can assume that $y \neq 0$. To separate variables, you must rid the first term of y and the second term of e^{-x^2}. So, you should multiply by e^{x^2}/y and obtain the following.

$$\begin{aligned} xy\,dx + e^{-x^2}(y^2 - 1)\,dy &= 0 \\ e^{-x^2}(y^2 - 1)\,dy &= -xy\,dx \\ \int\left(y - \frac{1}{y}\right)dy &= \int -xe^{x^2}\,dx \\ \frac{y^2}{2} - \ln|y| &= -\frac{1}{2}e^{x^2} + C \end{aligned}$$

From the initial condition $y(0) = 1$, you have $\frac{1}{2} - 0 = -\frac{1}{2} + C$, which implies that $C = 1$. So, the particular solution has the implicit form

$$\begin{aligned} \frac{y^2}{2} - \ln|y| &= -\frac{1}{2}e^{x^2} + 1 \\ y^2 - \ln y^2 + e^{x^2} &= 2. \end{aligned}$$

You can check this by differentiating and rewriting to get the original equation.

EXAMPLE 3 Finding a Particular Solution Curve

Find the equation of the curve that passes through the point $(1, 3)$ and has a slope of y/x^2 at any point (x, y).

Solution Because the slope of the curve is given by y/x^2, you have

$$\frac{dy}{dx} = \frac{y}{x^2}$$

with the initial condition $y(1) = 3$. Separating variables and integrating produces

$$\begin{aligned} \int\frac{dy}{y} &= \int\frac{dx}{x^2}, \quad y \neq 0 \\ \ln|y| &= -\frac{1}{x} + C_1 \\ y &= e^{-(1/x)+C_1} = Ce^{-1/x}. \end{aligned}$$

Because $y = 3$ when $x = 1$, it follows that $3 = Ce^{-1}$ and $C = 3e$. So, the equation of the specified curve is

$$y = (3e)e^{-1/x} = 3e^{(x-1)/x}, \quad x > 0.$$

See Figure 6.12.

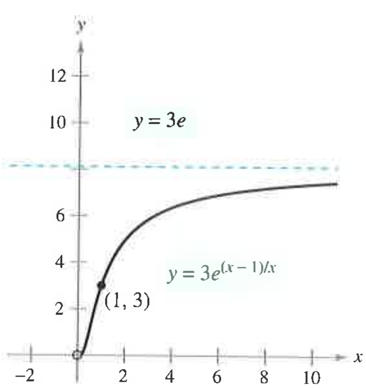


Figure 6.12

Homogeneous Differential Equations

Some differential equations that are not separable in x and y can be made separable by a change of variables. This is true for differential equations of the form $y' = f(x, y)$, where f is a **homogeneous function.** The function given by $f(x, y)$ is **homogeneous of degree n** if

NOTE The notation $f(x, y)$ is used to denote a function of two variables in much the same way as $f(x)$ denotes a function of one variable. You will study functions of two variables in detail in Chapter 13.

$$f(tx, ty) = t^n f(x, y)$$ Homogeneous function of degree n

where n is a real number.

EXAMPLE 4 Verifying Homogeneous Functions

a. $f(x, y) = x^2y - 4x^3 + 3xy^2$ is a homogeneous function of degree 3 because

$$\begin{aligned} f(tx, ty) &= (tx)^2(ty) - 4(tx)^3 + 3(tx)(ty)^2 \\ &= t^3(x^2y) - t^3(4x^3) + t^3(3xy^2) \\ &= t^3(x^2y - 4x^3 + 3xy^2) \\ &= t^3 f(x, y). \end{aligned}$$

b. $f(x, y) = xe^{x/y} + y\sin(y/x)$ is a homogeneous function of degree 1 because

$$\begin{aligned} f(tx, ty) &= txe^{tx/ty} + ty\sin\frac{ty}{tx} \\ &= t\left(xe^{x/y} + y\sin\frac{y}{x}\right) \\ &= tf(x, y). \end{aligned}$$

c. $f(x, y) = x + y^2$ is *not* a homogeneous function because

$$f(tx, ty) = tx + t^2y^2 = t(x + ty^2) \neq t^n(x + y^2).$$

d. $f(x, y) = x/y$ is a homogeneous function of degree 0 because

$$f(tx, ty) = \frac{tx}{ty} = t^0\frac{x}{y}.$$

Definition of Homogeneous Differential Equation

A **homogeneous differential equation** is an equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

where M and N are homogeneous functions of the same degree.

EXAMPLE 5 Testing for Homogeneous Differential Equations

a. $(x^2 + xy)\,dx + y^2\,dy = 0$ is homogeneous of degree 2.

b. $x^3\,dx = y^3\,dy$ is homogeneous of degree 3.

c. $(x^2 + 1)\,dx + y^2\,dy = 0$ is *not* a homogeneous differential equation.

To solve a homogeneous differential equation by the method of separation of variables, use the following change of variables theorem.

THEOREM 6.2 Change of Variables for Homogeneous Equations

If $M(x, y)\,dx + N(x, y)\,dy = 0$ is homogeneous, then it can be transformed into a differential equation whose variables are separable by the substitution

$$y = vx$$

where v is a differentiable function of x.

EXAMPLE 6 Solving a Homogeneous Differential Equation

Find the general solution of

$$(x^2 - y^2)\,dx + 3xy\,dy = 0.$$

STUDY TIP The substitution $y = vx$ will yield a differential equation that is separable with respect to the variables x and v. You must write your final solution, however, in terms of x and y.

Solution Because $(x^2 - y^2)$ and $3xy$ are both homogeneous of degree 2, let $y = vx$ to obtain $dy = x\,dv + v\,dx$. Then, by substitution, you have

$$\begin{aligned}(x^2 - v^2x^2)\,dx + 3x(vx)\overbrace{(x\,dv + v\,dx)}^{dy} &= 0\\ (x^2 + 2v^2x^2)\,dx + 3x^3v\,dv &= 0\\ x^2(1 + 2v^2)\,dx + x^2(3vx)\,dv &= 0.\end{aligned}$$

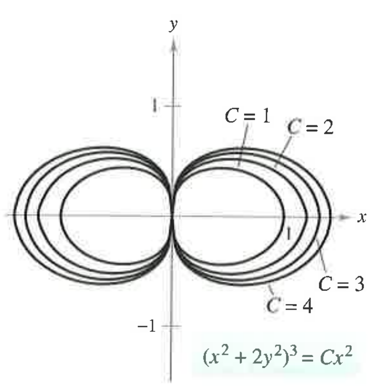
Dividing by x^2 and separating variables produces

$$\begin{aligned}(1 + 2v^2)\,dx &= -3vx\,dv\\ \int \frac{dx}{x} &= \int \frac{-3v}{1 + 2v^2}\,dv\\ \ln|x| &= -\frac{3}{4}\ln(1 + 2v^2) + C_1\\ 4\ln|x| &= -3\ln(1 + 2v^2) + \ln|C|\\ \ln x^4 &= \ln|C(1 + 2v^2)^{-3}|\\ x^4 &= C(1 + 2v^2)^{-3}.\end{aligned}$$

Substituting for v produces the following general solution.

$$\begin{aligned}x^4 &= C\left[1 + 2\left(\frac{y}{x}\right)^2\right]^{-3}\\ \left(1 + \frac{2y^2}{x^2}\right)^3 x^4 &= C\\ (x^2 + 2y^2)^3 &= Cx^2 \qquad \text{General solution}\end{aligned}$$

You can check this by differentiating and rewriting to get the original equation.



General solutions of
$(x^2 - y^2)\,dx + 3xy\,dy = 0$
Figure 6.13

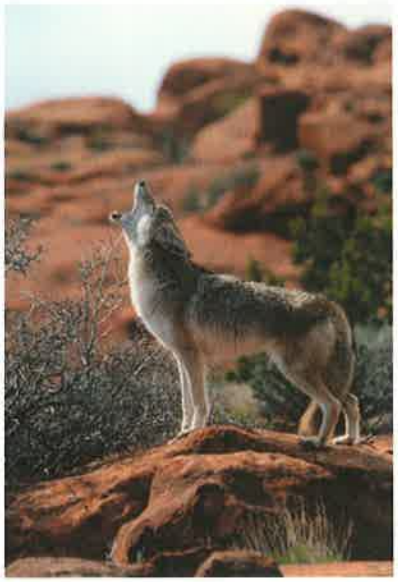
TECHNOLOGY If you have access to a graphing utility, try using it to graph several of the solutions in Example 6. For instance, Figure 6.13 shows the graphs of

$$(x^2 + 2y^2)^3 = Cx^2$$

for $C = 1, 2, 3,$ and 4.

Applications

EXAMPLE 7 Wildlife Population



The rate of change of the number of coyotes $N(t)$ in a population is directly proportional to $650 - N(t)$, where t is the time in years. When $t = 0$, the population is 300, and when $t = 2$, the population has increased to 500. Find the population when $t = 3$.

Solution Because the rate of change of the population is proportional to $650 - N(t)$, you can write the following differential equation.

$$\frac{dN}{dt} = k(650 - N)$$

You can solve this equation using separation of variables.

$$dN = k(650 - N)\,dt \qquad \text{Differential form}$$

$$\frac{dN}{650 - N} = k\,dt \qquad \text{Separate variables.}$$

$$-\ln|650 - N| = kt + C_1 \qquad \text{Integrate.}$$

$$\ln|650 - N| = -kt - C_1$$

$$650 - N = e^{-kt - C_1} \qquad \text{Assume } N < 650.$$

$$N = 650 - Ce^{-kt} \qquad \text{General solution}$$

Using $N = 300$ when $t = 0$, you can conclude that $C = 350$, which produces

$$N = 650 - 350e^{-kt}.$$

Then, using $N = 500$ when $t = 2$, it follows that

$$500 = 650 - 350e^{-2k} \quad \Rightarrow \quad e^{-2k} = \frac{3}{7} \quad \Rightarrow \quad k \approx 0.4236.$$

So, the model for the coyote population is

$$N = 650 - 350e^{-0.4236t}. \qquad \text{Model for population}$$

When $t = 3$, you can approximate the population to be

$$N = 650 - 350e^{-0.4236(3)} \approx 552 \text{ coyotes.}$$

The model for the population is shown in Figure 6.14.

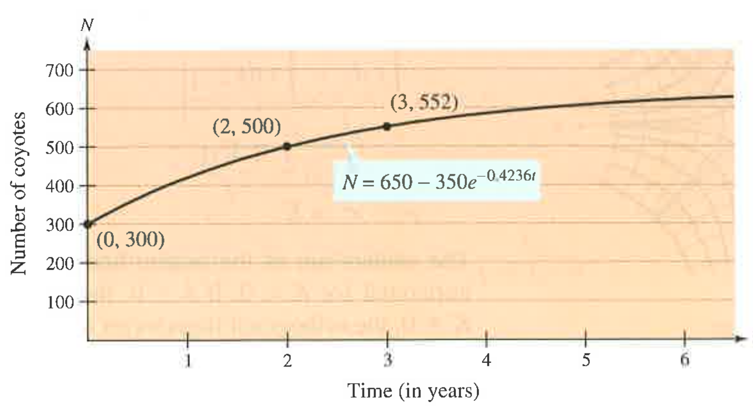


Figure 6.14

John Edwards/Getty Images

EXPLORATION

Explain what happens if $p(0) = L$.

EXAMPLE 10 Solving a Logistic Differential Equation

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population p is

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{4000}\right), \quad 40 \le p \le 4000$$

where t is the number of years.

a. Write a model for the elk population in terms of t.

b. Graph the slope field of the differential equation and the solution that passes through the point $(0, 40)$.

c. Use the model to estimate the elk population after 15 years.

d. Find the limit of the model as $t \to \infty$.

Solution

a. You know that $L = 4000$. So, the solution of the equation is of the form

$$p = \frac{4000}{1 + be^{-kt}}.$$

Because $p(0) = 40$, you can solve for b as shown.

$$40 = \frac{4000}{1 + be^{-k(0)}}$$

$$40 = \frac{4000}{1 + b} \quad \Rightarrow \quad b = 99$$

Then, because $p = 104$ when $t = 5$, you can solve for k.

$$104 = \frac{4000}{1 + 99e^{-k(5)}} \quad \Rightarrow \quad k \approx 0.194$$

So, a model for the elk population is given by $p = \dfrac{4000}{1 + 99e^{-0.194t}}$.

b. Using a graphing utility, you can graph the slope field of

$$\frac{dp}{dt} = 0.194p\left(1 - \frac{p}{4000}\right)$$

and the solution that passes through $(0, 40)$, as shown in Figure 6.18.

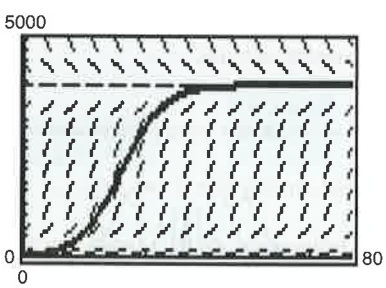


Figure 6.18

c. To estimate the elk population after 15 years, substitute 15 for t in the model

$$p = \frac{4000}{1 + 99e^{-0.194(15)}} \qquad \text{Substitute 15 for } t.$$

$$= \frac{4000}{1 + 99e^{-2.91}} \approx 626 \qquad \text{Simplify.}$$

d. As t increases without bound, the denominator of $\dfrac{4000}{1 + 99e^{-0.194t}}$ gets closer to 1.

So, $\displaystyle\lim_{t \to \infty} \frac{4000}{1 + 99e^{-0.194t}} = 4000.$

Exercises for Section 6.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–12, find the general solution of the differential equation.

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$
2. $\dfrac{dy}{dx} = \dfrac{x^2 + 2}{3y^2}$
3. $\dfrac{dr}{ds} = 0.05r$
4. $\dfrac{dr}{ds} = 0.05s$
5. $(2 + x)y' = 3y$
6. $xy' = y$
7. $yy' = \sin x$
8. $yy' = 6\cos(\pi x)$
9. $\sqrt{1 - 4x^2}\, y' = x$
10. $\sqrt{x^2 - 9}\, y' = 5x$
11. $y \ln x - xy' = 0$
12. $4yy' - 3e^x = 0$

In Exercises 13–22, find the particular solution that satisfies the initial condition.

	Differential Equation	*Initial Condition*
13.	$yy' - e^x = 0$	$y(0) = 4$
14.	$\sqrt{x} + \sqrt{y}\, y' = 0$	$y(1) = 4$
15.	$y(x + 1) + y' = 0$	$y(-2) = 1$
16.	$2xy' - \ln x^2 = 0$	$y(1) = 2$
17.	$y(1 + x^2)y' - x(1 + y^2) = 0$	$y(0) = \sqrt{3}$
18.	$y\sqrt{1 - x^2}\, y' - x\sqrt{1 - y^2} = 0$	$y(0) = 1$
19.	$\dfrac{du}{dv} = uv \sin v^2$	$u(0) = 1$
20.	$\dfrac{dr}{ds} = e^{r-2s}$	$r(0) = 0$
21.	$dP - kP\,dt = 0$	$P(0) = P_0$
22.	$dT + k(T - 70)\,dt = 0$	$T(0) = 140$

In Exercises 23 and 24, find an equation of the graph that passes through the point and has the given slope.

23. $(1, 1),\quad y' = -\dfrac{9x}{16y}$
24. $(8, 2),\quad y' = \dfrac{2y}{3x}$

In Exercises 25 and 26, find all functions f having the indicated property.

25. The tangent to the graph of f at the point (x, y) intersects the x-axis at $(x + 2, 0)$.
26. All tangents to the graph of f pass through the origin.

In Exercises 27–34, determine whether the function is homogeneous, and if it is, determine its degree.

27. $f(x, y) = x^3 - 4xy^2 + y^3$
28. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$
29. $f(x, y) = \dfrac{x^2y^2}{\sqrt{x^2 + y^2}}$
30. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$
31. $f(x, y) = 2 \ln xy$
32. $f(x, y) = \tan(x + y)$
33. $f(x, y) = 2 \ln \dfrac{x}{y}$
34. $f(x, y) = \tan \dfrac{y}{x}$

In Exercises 35–40, solve the homogeneous differential equation.

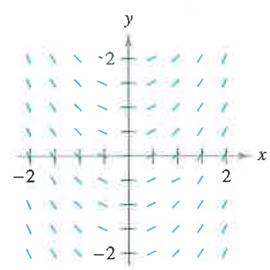
35. $y' = \dfrac{x + y}{2x}$
36. $y' = \dfrac{x^3 + y^3}{xy^2}$
37. $y' = \dfrac{x - y}{x + y}$
38. $y' = \dfrac{x^2 + y^2}{2xy}$
39. $y' = \dfrac{xy}{x^2 - y^2}$
40. $y' = \dfrac{2x + 3y}{x}$

In Exercises 41–44, find the particular solution that satisfies the initial condition.

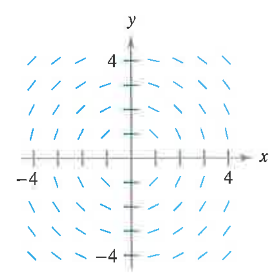
	Differential Equation	*Initial Condition*
41.	$x\,dy - (2xe^{-y/x} + y)\,dx = 0$	$y(1) = 0$
42.	$-y^2\,dx + x(x + y)\,dy = 0$	$y(1) = 1$
43.	$\left(x \sec \dfrac{y}{x} + y\right) dx - x\,dy = 0$	$y(1) = 0$
44.	$(2x^2 + y^2)\,dx + xy\,dy = 0$	$y(1) = 0$

Slope Fields **In Exercises 45–48, sketch a few solutions of the differential equation on the slope field and then find the general solution analytically. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

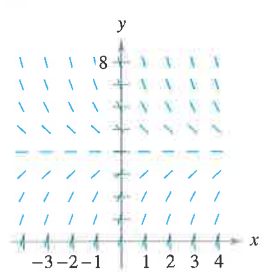
45. $\dfrac{dy}{dx} = x$



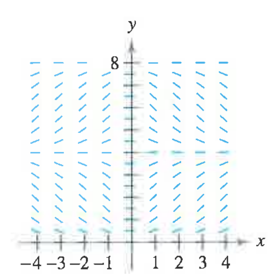
46. $\dfrac{dy}{dx} = -\dfrac{x}{y}$



47. $\dfrac{dy}{dx} = 4 - y$



48. $\dfrac{dy}{dx} = 0.25x(4 - y)$



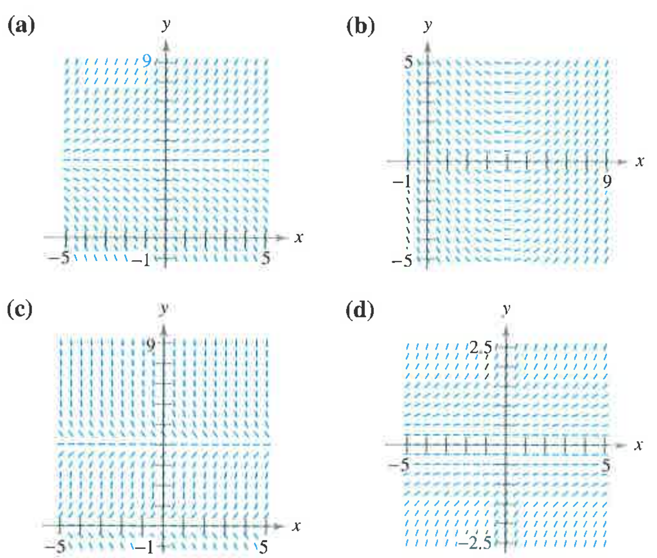
Euler's Method **In Exercises 49–52, (a) use Euler's Method with a step size of $h = 0.1$ to approximate the particular solution of the initial value problem at the given x-value, (b) find the exact solution of the differential equation analytically, and (c) compare the solutions at the given x-value.**

	Differential Equation	*Initial Condition*	*x-value*
49.	$\frac{dy}{dx} = -6xy$	$(0, 5)$	$x = 1$
50.	$\frac{dy}{dx} + 6xy^2 = 0$	$(0, 3)$	$x = 1$
51.	$\frac{dy}{dx} = \frac{2x + 12}{3y^2 - 4}$	$(1, 2)$	$x = 2$
52.	$\frac{dy}{dx} = 2x(1 + y^2)$	$(1, 0)$	$x = 1.5$

53. ***Radioactive Decay*** The rate of decomposition of radioactive radium is proportional to the amount present at any time. The half-life of radioactive radium is 1599 years. What percent of a present amount will remain after 25 years?

54. ***Chemical Reaction*** In a chemical reaction, a certain compound changes into another compound at a rate proportional to the unchanged amount. If initially there are 20 grams of the original compound, and there is 16 grams after 1 hour, when will 75 percent of the compound be changed?

Slope Fields **In Exercises 55–58, (a) write a differential equation for the statement, (b) match the differential equation with a possible slope field, and (c) verify your result by using a graphing utility to graph a slope field for the differential equation. [The slope fields are labeled (a), (b), (c), and (d).] To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**



55. The rate of change of y with respect to x is proportional to the difference between y and 4.

56. The rate of change of y with respect to x is proportional to the difference between x and 4.

57. The rate of change of y with respect to x is proportional to the product of y and the difference between y and 4.

58. The rate of change of y with respect to x is proportional to y^2.

59. ***Weight Gain*** A calf that weighs 60 pounds at birth gains weight at the rate

$$\frac{dw}{dt} = k(1200 - w)$$

where w is weight in pounds and t is time in years. Solve the differential equation.

(a) Use a computer algebra system to solve the differential equation for $k = 0.8, 0.9$, and 1. Graph the three solutions.

(b) If the animal is sold when its weight reaches 800 pounds, find the time of sale for each of the models in part (a).

(c) What is the maximum weight of the animal for each of the models?

60. ***Weight Gain*** A calf that weighs w_0 pounds at birth gains weight at the rate

$$\frac{dw}{dt} = 1200 - w$$

where w is weight in pounds and t is time in years. Solve the differential equation.

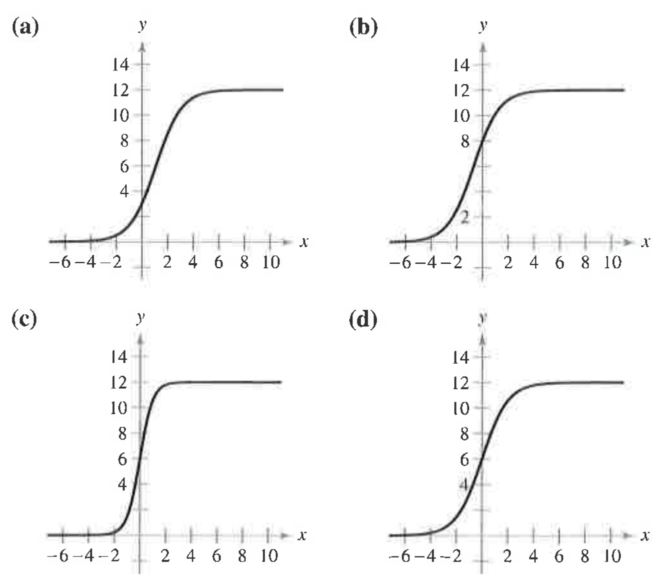
In Exercises 61–66, find the orthogonal trajectories of the family. Use a graphing utility to graph several members of each family.

61. $x^2 + y^2 = C$ **62.** $x^2 - 2y^2 = C$

63. $x^2 = Cy$ **64.** $y^2 = 2Cx$

65. $y^2 = Cx^3$ **66.** $y = Ce^x$

In Exercises 67–70, match the logistic equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]



67. $y = \dfrac{12}{1 + e^{-x}}$

68. $y = \dfrac{12}{1 + 3e^{-x}}$

69. $y = \dfrac{12}{1 + \frac{1}{2}e^{-x}}$

70. $y = \dfrac{12}{1 + e^{-2x}}$

In Exercises 71 and 72, the logistic equation models the growth of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution $P(t)$.

71. $P(t) = \dfrac{1500}{1 + 24e^{-0.75t}}$

72. $P(t) = \dfrac{5000}{1 + 39e^{-0.2t}}$

In Exercises 73 and 74, the logistic differential equation models the growth rate of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) use a computer algebra system to graph a slope field, and (d) determine the value of P at which the population growth rate is the greatest.

73. $\dfrac{dP}{dt} = 3P\left(1 - \dfrac{P}{100}\right)$

74. $\dfrac{dP}{dt} = 0.1P - 0.0004P^2$

In Exercises 75–78, find the logistic equation that satisfies the initial condition.

Logistic Differential Equation	*Initial Condition*
75. $\dfrac{dy}{dt} = y\left(1 - \dfrac{y}{40}\right)$	(0, 8)
76. $\dfrac{dy}{dt} = 1.2y\left(1 - \dfrac{y}{8}\right)$	(0, 5)
77. $\dfrac{dy}{dt} = \dfrac{4y}{5} - \dfrac{y^2}{150}$	(0, 8)
78. $\dfrac{dy}{dt} = \dfrac{3y}{20} - \dfrac{y^2}{1600}$	(0, 15)

79. ***Endangered Species*** A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. The Florida preserve has a carrying capacity of 200 panthers.

(a) Write a logistic equation that models the population of panthers in the preserve.

(b) Find the population after 5 years.

(c) When will the population reach 100?

(d) Write a logistic differential equation that models the growth rate of the panther population. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answers.

(e) At what time is the panther population growing most rapidly? Explain.

80. ***Bacteria Growth*** At time $t = 0$, a bacterial culture weighs 1 gram. Two hours later, the culture weighs 2 grams. The maximum weight of the culture is 10 grams.

(a) Write a logistic equation that models the weight of the bacterial culture.

(b) Find the culture's weight after 5 hours.

(c) When will the culture's weight reach 8 grams?

(d) Write a logistic differential equation that models the growth rate of the culture's weight. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answers.

(e) At what time is the culture's weight increasing most rapidly? Explain.

Writing About Concepts

81. In your own words, describe how to recognize and solve differential equations that can be solved by separation of variables.

82. State the test for determining if a differential equation is homogeneous. Give an example.

83. In your own words, describe the relationship between two families of curves that are mutually orthogonal.

84. ***Sailing*** Ignoring resistance, a sailboat starting from rest accelerates (dv/dt) at a rate proportional to the difference between the velocities of the wind and the boat.

(a) The wind is blowing at 20 knots, and after 1 minute the boat is moving at 5 knots. Write the velocity v as a function of time t.

(b) Use the result of part (a) to write the distance traveled by the boat as a function of time.

True or False? **In Exercises 85–88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

85. The function $y = 0$ is always a solution of a differential equation that can be solved by separation of variables.

86. The differential equation $y' = xy - 2y + x - 2$ can be written in separated variables form.

87. The function $f(x, y) = x^2 + xy + 2$ is homogeneous.

88. The families $x^2 + y^2 = 2Cy$ and $x^2 + y^2 = 2Kx$ are mutually orthogonal.

89. Show that if $y = \dfrac{1}{1 + be^{-kt}}$, then $\dfrac{dy}{dt} = ky(1 - y)$.

Putnam Exam Challenge

90. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

This problem was composed by the Committee on the Putnam Prize Competition.

Section 6.4

First-Order Linear Differential Equations

- **Solve a first-order linear differential equation.**
- **Solve a Bernoulli differential equation.**
- **Use linear differential equations to solve applied problems.**

First-Order Linear Differential Equations

In this section, you will see how to solve a very important class of first-order differential equations—first-order linear differential equations.

Definition of First-Order Linear Differential Equation

A first-order linear differential equation is an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions of x. This first-order linear differential equation is said to be in **standard form.**

NOTE It is instructive to see why the integrating factor helps solve a linear differential equation of the form $y' + P(x)y = Q(x)$. When both sides of the equation are multiplied by the integrating factor $u(x) = e^{\int P(x)\,dx}$, the left-hand side becomes the derivative of a product.

$$y'e^{\int P(x)\,dx} + P(x)ye^{\int P(x)\,dx} = Q(x)e^{\int P(x)\,dx}$$
$$\left[ye^{\int P(x)\,dx}\right]' = Q(x)e^{\int P(x)\,dx}$$

Integrating both sides of this second equation and dividing by $u(x)$ produces the general solution.

To solve a linear differential equation, write it in standard form to identify the functions $P(x)$ and $Q(x)$. Then integrate $P(x)$ and form the expression

$$u(x) = e^{\int P(x)\,dx} \qquad \text{Integrating factor}$$

which is called an **integrating factor.** The general solution of the equation is

$$y = \frac{1}{u(x)}\int Q(x)u(x)\,dx. \qquad \text{General solution}$$

EXAMPLE 1 Solving a Linear Differential Equation

Find the general solution of

$$y' + y = e^x.$$

Solution

For this equation, $P(x) = 1$ and $Q(x) = e^x$. So, the integrating factor is

$$\begin{aligned} u(x) &= e^{\int P(x)\,dx} && \text{Integrating factor} \\ &= e^{\int dx} \\ &= e^x. \end{aligned}$$

This implies that the general solution is

$$\begin{aligned} y &= \frac{1}{u(x)}\int Q(x)u(x)\,dx \\ &= \frac{1}{e^x}\int e^x(e^x)\,dx \\ &= e^{-x}\left(\frac{1}{2}e^{2x} + C\right) \\ &= \frac{1}{2}e^x + Ce^{-x}. && \text{General solution} \end{aligned}$$

THEOREM 6.3 Solution of a First-Order Linear Differential Equation

An integrating factor for the first-order linear differential equation

$$y' + P(x)y = Q(x)$$

is $u(x) = e^{\int P(x)\,dx}$. The solution of the differential equation is

$$ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)\,dx}\,dx + C.$$

ANNA JOHNSON PELL WHEELER (1883–1966)

Anna Johnson Pell Wheeler was awarded a master's degree from the University of Iowa for her thesis *The Extension of Galois Theory to Linear Differential Equations* in 1904. Influenced by David Hilbert, she worked on integral equations while studying infinite linear spaces.

STUDY TIP Rather than memorizing the formula in Theorem 6.3, just remember that multiplication by the integrating factor $e^{\int P(x)\,dx}$ converts the left side of the differential equation into the derivative of the product $ye^{\int P(x)\,dx}$.

EXAMPLE 2 Solving a First-Order Linear Differential Equation

Find the general solution of

$$xy' - 2y = x^2.$$

Solution The standard form of the given equation is

$$y' + P(x)y = Q(x)$$

$$y' - \left(\frac{2}{x}\right)y = x. \qquad \text{Standard form}$$

So, $P(x) = -2/x$, and you have

$$\int P(x)\,dx = -\int \frac{2}{x}\,dx$$

$$= -\ln x^2$$

$$e^{\int P(x)\,dx} = e^{-\ln x^2}$$

$$= \frac{1}{e^{\ln x^2}}$$

$$= \frac{1}{x^2}. \qquad \text{Integrating factor}$$

So, multiplying each side of the standard form by $1/x^2$ yields

$$\frac{y'}{x^2} - \frac{2y}{x^3} = \frac{1}{x}$$

$$\frac{d}{dx}\left[\frac{y}{x^2}\right] = \frac{1}{x}$$

$$\frac{y}{x^2} = \int \frac{1}{x}\,dx$$

$$\frac{y}{x^2} = \ln|x| + C$$

$$y = x^2(\ln|x| + C). \qquad \text{General solution}$$

Several solution curves (for $C = -2, -1, 0, 1, 2, 3,$ and 4) are shown in Figure 6.19.

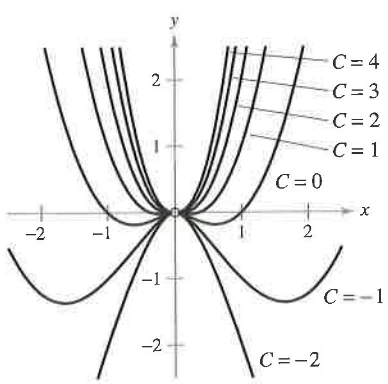


Figure 6.19

EXAMPLE 3 Solving a First-Order Linear Differential Equation

Find the general solution of

$$y' - y\tan t = 1, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Solution The equation is already in the standard form $y' + P(t)y = Q(t)$. So, $P(t) = -\tan t$, and

$$\int P(t)\,dt = -\int \tan t\,dt = \ln|\cos t|$$

which implies that the integrating factor is

$$e^{\int P(t)\,dt} = e^{\ln|\cos t|}$$
$$= |\cos t|. \qquad \text{Integrating factor}$$

A quick check shows that $\cos t$ is also an integrating factor. So, multiplying $y' - y\tan t = 1$ by $\cos t$ produces

$$\frac{d}{dt}[y\cos t] = \cos t$$
$$y\cos t = \int \cos t\,dt$$
$$y\cos t = \sin t + C$$
$$y = \tan t + C\sec t. \qquad \text{General solution}$$

Several solution curves are shown in Figure 6.20.

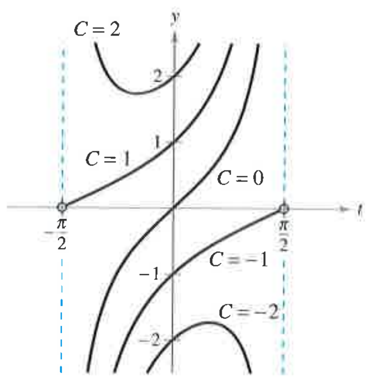


Figure 6.20

Bernoulli Equation

A well-known nonlinear equation that reduces to a linear one with an appropriate substitution is the **Bernoulli equation,** named after James Bernoulli (1654–1705).

$$y' + P(x)y = Q(x)y^n \qquad \text{Bernoulli equation}$$

This equation is linear if $n = 0$, and has separable variables if $n = 1$. So, in the following development, assume that $n \neq 0$ and $n \neq 1$. Begin by multiplying by y^{-n} and $(1 - n)$ to obtain

$$y^{-n}y' + P(x)y^{1-n} = Q(x)$$
$$(1-n)y^{-n}y' + (1-n)P(x)y^{1-n} = (1-n)Q(x)$$
$$\frac{d}{dx}[y^{1-n}] + (1-n)P(x)y^{1-n} = (1-n)Q(x)$$

which is a linear equation in the variable y^{1-n}. Letting $z = y^{1-n}$ produces the linear equation

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).$$

Finally, by Theorem 6.3, the *general solution of the Bernoulli equation* is

$$y^{1-n}e^{\int(1-n)P(x)\,dx} = \int (1-n)Q(x)e^{\int(1-n)P(x)\,dx}\,dx + C.$$

EXAMPLE 4 Solving a Bernoulli Equation

Find the general solution of

$$y' + xy = xe^{-x^2}y^{-3}.$$

Solution For this Bernoulli equation, let $n = -3$, and use the substitution

$$z = y^4 \qquad \text{Let } z = y^{1-n} = y^{1-(-3)}.$$

$$z' = 4y^3y'. \qquad \text{Differentiate.}$$

Multiplying the original equation by $4y^3$ produces

$$y' + xy = xe^{-x^2}y^{-3} \qquad \text{Write original equation.}$$

$$4y^3y' + 4xy^4 = 4xe^{-x^2} \qquad \text{Multiply each side by } 4y^3.$$

$$z' + 4xz = 4xe^{-x^2}. \qquad \text{Linear equation: } z' + P(x)z = Q(x)$$

This equation is linear in z. Using $P(x) = 4x$ produces

$$\int P(x)\,dx = \int 4x\,dx$$

$$= 2x^2$$

which implies that e^{2x^2} is an integrating factor. Multiplying the linear equation by this factor produces

$$z' + 4xz = 4xe^{-x^2} \qquad \text{Linear equation}$$

$$z'e^{2x^2} + 4xze^{2x^2} = 4xe^{x^2} \qquad \text{Multiply by integrating factor.}$$

$$\frac{d}{dx}[ze^{2x^2}] = 4xe^{x^2} \qquad \text{Write left side as derivative.}$$

$$ze^{2x^2} = \int 4xe^{x^2}\,dx \qquad \text{Integrate each side.}$$

$$ze^{2x^2} = 2e^{x^2} + C$$

$$z = 2e^{-x^2} + Ce^{-2x^2}. \qquad \text{Divide each side by } e^{2x^2}.$$

Finally, substituting $z = y^4$, the general solution is

$$y^4 = 2e^{-x^2} + Ce^{-2x^2}. \qquad \text{General solution}$$

So far you have studied several types of first-order differential equations. Of these, the separable variables case is usually the simplest, and solution by an integrating factor is ordinarily used only as a last resort.

Summary of First-Order Differential Equations

Method	*Form of Equation*
1. Separable variables:	$M(x)\,dx + N(y)\,dy = 0$
2. Homogeneous:	$M(x, y)\,dx + N(x, y)\,dy = 0$, where M and N are nth-degree homogeneous
3. Linear:	$y' + P(x)y = Q(x)$
4. Bernoulli equation:	$y' + P(x)y = Q(x)y^n$

Applications

One type of problem that can be described in terms of a differential equation involves chemical mixtures, as illustrated in the next example.

EXAMPLE 5 A Mixture Problem

Figure 6.21

A tank contains 50 gallons of a solution composed of 90% water and 10% alcohol. A second solution containing 50% water and 50% alcohol is added to the tank at the rate of 4 gallons per minute. As the second solution is being added, the tank is being drained at a rate of 5 gallons per minute, as shown in Figure 6.21. Assuming the solution in the tank is stirred constantly, how much alcohol is in the tank after 10 minutes?

Solution Let y be the number of gallons of alcohol in the tank at any time t. You know that $y = 5$ when $t = 0$. Because the number of gallons of solution in the tank at any time is $50 - t$, and the tank loses 5 gallons of solution per minute, it must lose

$$\left(\frac{5}{50-t}\right)y$$

gallons of alcohol per minute. Furthermore, because the tank is gaining 2 gallons of alcohol per minute, the rate of change of alcohol in the tank is given by

$$\frac{dy}{dt} = 2 - \left(\frac{5}{50-t}\right)y \quad \Rightarrow \quad \frac{dy}{dt} + \left(\frac{5}{50-t}\right)y = 2.$$

To solve this linear equation, let $P(t) = 5/(50 - t)$ and obtain

$$\int P(t)\,dt = \int \frac{5}{50-t}\,dt = -5\ln|50-t|.$$

Because $t < 50$, you can drop the absolute value signs and conclude that

$$e^{\int P(t)\,dt} = e^{-5\ln(50-t)} = \frac{1}{(50-t)^5}.$$

So, the general solution is

$$\frac{y}{(50-t)^5} = \int \frac{2}{(50-t)^5}\,dt = \frac{1}{2(50-t)^4} + C$$

$$y = \frac{50-t}{2} + C(50-t)^5.$$

Because $y = 5$ when $t = 0$, you have

$$5 = \frac{50}{2} + C(50)^5 \quad \Rightarrow \quad -\frac{20}{50^5} = C$$

which means that the particular solution is

$$y = \frac{50-t}{2} - 20\left(\frac{50-t}{50}\right)^5.$$

Finally, when $t = 10$, the amount of alcohol in the tank is

$$y = \frac{50-10}{2} - 20\left(\frac{50-10}{50}\right)^5 \approx 13.45 \text{ gal}$$

which represents a solution containing 33.6% alcohol.

In most falling-body problems discussed so far in the text, air resistance has been neglected. The next example includes this factor. In the example, the air resistance on the falling object is assumed to be proportional to its velocity v. If g is the gravitational constant, the downward force F on a falling object of mass m is given by the difference $mg - kv$. But by Newton's Second Law of Motion, you know that

$$\begin{aligned} F &= ma \\ &= m(dv/dt) \end{aligned}$$

which yields the following differential equation.

$$m\frac{dv}{dt} = mg - kv \quad \Rightarrow \quad \frac{dv}{dt} + \frac{k}{m}v = g$$

EXAMPLE 6 A Falling Object with Air Resistance

An object of mass m is dropped from a hovering helicopter. Find its velocity as a function of time t, assuming that the air resistance is proportional to the velocity of the object.

Solution The velocity v satisfies the equation

$$\frac{dv}{dt} + \frac{kv}{m} = g$$

where g is the gravitational constant and k is the constant of proportionality. Letting $b = k/m$, you can *separate variables* to obtain

$$\begin{aligned} dv &= (g - bv)\,dt \\ \int \frac{dv}{g - bv} &= \int dt \\ -\frac{1}{b}\ln|g - bv| &= t + C_1 \\ \ln|g - bv| &= -bt - bC_1 \\ g - bv &= Ce^{-bt}. \end{aligned}$$

Because the object was dropped, $v = 0$ when $t = 0$; so $g = C$, and it follows that

$$-bv = -g + ge^{-bt} \quad \Rightarrow \quad v = \frac{g - ge^{-bt}}{b} = \frac{mg}{k}(1 - e^{-kt/m}).$$

NOTE Notice in Example 6 that the velocity approaches a limit of mg/k as a result of the air resistance. For falling-body problems in which air resistance is neglected, the velocity increases without bound.

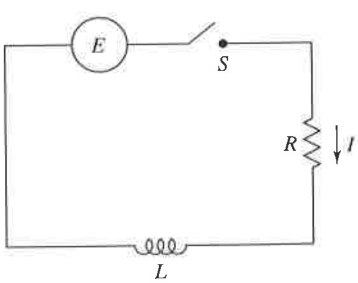


Figure 6.22

A simple electric circuit consists of electric current I (in amperes), a resistance R (in ohms), an inductance L (in henrys), and a constant electromotive force E (in volts), as shown in Figure 6.22. According to Kirchhoff's Second Law, if the switch S is closed when $t = 0$, the applied electromotive force (voltage) is equal to the sum of the voltage drops in the rest of the circuit. This in turn means that the current I satisfies the differential equation

$$L\frac{dI}{dt} + RI = E.$$

EXAMPLE 7 An Electric Circuit Problem

Find the current I as a function of time t (in seconds), given that I satisfies the differential equation $L(dI/dt) + RI = \sin 2t$, where R and L are nonzero constants.

TECHNOLOGY The integral in Example 7 was found using symbolic algebra software. If you have access to *Derive*, *Maple*, *Mathcad*, *Mathematica*, or the *TI-89*, try using it to integrate

$$\frac{1}{L}\int e^{(R/L)t}\sin 2t\,dt.$$

In Chapter 8 you will learn how to integrate functions of this type using integration by parts.

Solution In standard form, the given linear equation is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}\sin 2t.$$

Let $P(t) = R/L$, so that $e^{\int P(t)\,dt} = e^{(R/L)t}$, and, by Theorem 6.3,

$$Ie^{(R/L)t} = \frac{1}{L}\int e^{(R/L)t}\sin 2t\,dt$$
$$= \frac{1}{4L^2 + R^2}e^{(R/L)t}(R\sin 2t - 2L\cos 2t) + C.$$

So the general solution is

$$I = e^{-(R/L)t}\left[\frac{1}{4L^2 + R^2}e^{(R/L)t}(R\sin 2t - 2L\cos 2t) + C\right]$$
$$I = \frac{1}{4L^2 + R^2}(R\sin 2t - 2L\cos 2t) + Ce^{-(R/L)t}.$$

Exercises for Section 6.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, determine whether the differential equation is linear. Explain your reasoning.

1. $x^3y' + xy = e^x + 1$
2. $2xy - y'\ln x = y$
3. $y' + y\cos x = xy^2$
4. $\dfrac{1 - y'}{y} = 3x$

In Exercises 5–14, solve the first-order linear differential equation.

5. $\dfrac{dy}{dx} + \left(\dfrac{1}{x}\right)y = 3x + 4$
6. $\dfrac{dy}{dx} + \left(\dfrac{2}{x}\right)y = 3x + 2$
7. $y' - y = 10$
8. $y' + 2xy = 4x$
9. $(y + 1)\cos x\,dx - dy = 0$
10. $(y - 1)\sin x\,dx - dy = 0$
11. $(x - 1)y' + y = x^2 - 1$
12. $y' + 3y = e^{3x}$
13. $y' - 3x^2y = e^{x^3}$
14. $y' - y = \cos x$

Slope Fields **In Exercises 15 and 16, (a) sketch an approximate solution of the differential equation satisfying the initial condition by hand on the slope field, (b) find the particular solution that satisfies the initial condition, and (c) use a graphing utility to graph the particular solution. Compare the graph with the hand-drawn graph of part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

Differential Equation	*Initial Condition*
15. $\dfrac{dy}{dx} = e^x - y$	$(0, 1)$
16. $y' + \left(\dfrac{1}{x}\right)y = \sin x^2$	$(\sqrt{\pi}, 0)$

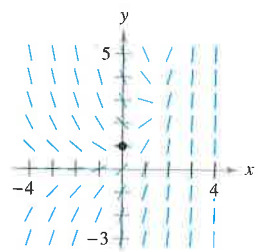


Figure for 15

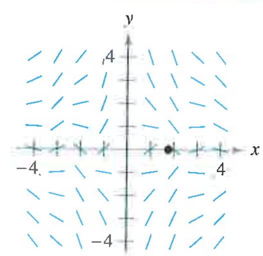


Figure for 16

In Exercises 17–24, find the particular solution of the differential equation that satisfies the boundary condition.

Differential Equation	*Boundary Condition*
17. $y'\cos^2 x + y - 1 = 0$	$y(0) = 5$
18. $x^3y' + 2y = e^{1/x^2}$	$y(1) = e$
19. $y' + y\tan x = \sec x + \cos x$	$y(0) = 1$
20. $y' + y\sec x = \sec x$	$y(0) = 4$
21. $y' + \left(\dfrac{1}{x}\right)y = 0$	$y(2) = 2$
22. $y' + (2x - 1)y = 0$	$y(1) = 2$
23. $x\,dy = (x + y + 2)\,dx$	$y(1) = 10$
24. $2xy' - y = x^3 - x$	$y(4) = 2$

In Exercises 25–30, solve the Bernoulli differential equation.

25. $y' + 3x^2y = x^2y^3$

26. $y' + xy = xy^{-1}$

27. $y' + \left(\frac{1}{x}\right)y = xy^2$

28. $y' + \left(\frac{1}{x}\right)y = x\sqrt{y}$

29. $y' - y = e^x\sqrt[3]{y}$

30. $yy' - 2y^2 = e^x$

Slope Fields **In Exercises 31–34, (a) use a graphing utility to graph the slope field for the differential equation, (b) find the particular solutions of the differential equation passing through the given points, and (c) use a graphing utility to graph the particular solutions on the slope field.**

Differential Equation	*Points*
31. $\dfrac{dy}{dx} - \dfrac{1}{x}y = x^2$	$(-2, 4),\ (2, 8)$
32. $\dfrac{dy}{dx} + 4x^3y = x^3$	$\left(0, \frac{7}{2}\right),\ \left(0, -\frac{1}{2}\right)$
33. $\dfrac{dy}{dx} + (\cot x)y = 2$	$(1, 1),\ (3, -1)$
34. $\dfrac{dy}{dx} + 2xy = xy^2$	$(0, 3),\ (0, 1)$

35. ***Population Growth*** When predicting population growth, demographers must consider birth and death rates as well as the net change caused by the difference between the rates of immigration and emigration. Let P be the population at time t and let N be the net increase per unit time resulting from the difference between immigration and emigration. So, the rate of growth of the population is given by

$$\frac{dP}{dt} = kP + N, \qquad N \text{ is constant.}$$

Solve this differential equation to find P as a function of time if at time $t = 0$ the size of the population is P_0.

36. ***Investment Growth*** A large corporation starts at time $t = 0$ to invest part of its receipts continuously at a rate of P dollars per year in a fund for future corporate expansion. Assume that the fund earns r percent interest per year compounded continuously. So, the rate of growth of the amount A in the fund is given by

$$\frac{dA}{dt} = rA + P$$

where $A = 0$ when $t = 0$. Solve this differential equation for A as a function of t.

Investment Growth **In Exercises 37 and 38, use the result of Exercise 36.**

37. Find A for the following.

(a) $P = \$100{,}000$, $r = 6\%$, and $t = 5$ years

(b) $P = \$250{,}000$, $r = 5\%$, and $t = 10$ years

38. Find t if the corporation needs \$800,000 and it can invest \$75,000 per year in a fund earning 8% interest compounded continuously.

39. ***Intravenous Feeding*** Glucose is added intravenously to the bloodstream at the rate of q units per minute, and the body removes glucose from the bloodstream at a rate proportional to the amount present. Assume that $Q(t)$ is the amount of glucose in the bloodstream at time t.

(a) Determine the differential equation describing the rate of change of glucose in the bloodstream with respect to time.

(b) Solve the differential equation from part (a), letting $Q = Q_0$ when $t = 0$.

(c) Find the limit of $Q(t)$ as $t \to \infty$.

40. ***Learning Curve*** The management at a certain factory has found that the maximum number of units a worker can produce in a day is 40. The rate of increase in the number of units N produced with respect to time t in days by a new employee is proportional to $40 - N$.

(a) Determine the differential equation describing the rate of change of performance with respect to time.

(b) Solve the differential equation from part (a).

(c) Find the particular solution for a new employee who produced 10 units on the first day at the factory and 19 units on the twentieth day.

Mixture **In Exercises 41–46, consider a tank that at time $t = 0$ contains v_0 gallons of a solution of which, by weight, q_0 pounds is soluble concentrate. Another solution containing q_1 pounds of the concentrate per gallon is running into the tank at the rate of r_1 gallons per minute. The solution in the tank is kept well stirred and is withdrawn at the rate of r_2 gallons per minute.**

41. If Q is the amount of concentrate in the solution at any time t, show that

$$\frac{dQ}{dt} + \frac{r_2Q}{v_0 + (r_1 - r_2)t} = q_1r_1.$$

42. If Q is the amount of concentrate in the solution at any time t, write the differential equation for the rate of change of Q with respect to t if $r_1 = r_2 = r$.

43. A 200-gallon tank is full of a solution containing 25 pounds of concentrate. Starting at time $t = 0$, distilled water is admitted to the tank at a rate of 10 gallons per minute, and the well-stirred solution is withdrawn at the same rate.

(a) Find the amount of concentrate Q in the solution as a function of t.

(b) Find the time at which the amount of concentrate in the tank reaches 15 pounds.

(c) Find the quantity of the concentrate in the solution as $t \to \infty$.

44. Repeat Exercise 43, assuming that the solution entering the tank contains 0.04 pound of concentrate per gallon.

45. A 200-gallon tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 pound of concentrate per gallon enters the tank at the rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at the rate of 3 gallons per minute.

(a) At what time will the tank be full?

(b) At the time the tank is full, how many pounds of concentrate will it contain?

46. Repeat Exercise 45, assuming that the solution entering the tank contains 1 pound of concentrate per gallon.

Falling Object **In Exercises 47 and 48, consider an eight-pound object dropped from a height of 5000 feet, where the air resistance is proportional to the velocity.**

47. Write the velocity as a function of time if its velocity after 5 seconds is approximately -101 feet per second. What is the limiting value of the velocity function?

48. Use the result of Exercise 47 to write the position of the object as a function of time. Approximate the velocity of the object when it reaches ground level.

Electric Circuits **In Exercises 49 and 50, use the differential equation for electric circuits given by**

$$L\frac{dI}{dt} + RI = E.$$

In this equation, I is the current, R is the resistance, L is the inductance, and E is the electromotive force (voltage).

49. Solve the differential equation given a constant voltage E_0.

50. Use the result of Exercise 49 to find the equation for the current if $I(0) = 0$, $E_0 = 120$ volts, $R = 600$ ohms, and $L = 4$ henrys. When does the current reach 90% of its limiting value?

Writing About Concepts

51. Give the standard form of a first-order linear differential equation. What is its integrating factor?

52. Give the standard form of the Bernoulli equation. Describe how one reduces it to a linear equation.

In Exercises 53–56, match the differential equation with its solution.

Differential Equation	*Solution*
53. $y' - 2x = 0$	(a) $y = Ce^{x^2}$
54. $y' - 2y = 0$	(b) $y = -\frac{1}{2} + Ce^{x^2}$
55. $y' - 2xy = 0$	(c) $y = x^2 + C$
56. $y' - 2xy = x$	(d) $y = Ce^{2x}$

In Exercises 57–68, solve the first-order differential equation by any appropriate method.

57. $\dfrac{dy}{dx} = \dfrac{e^{2x+y}}{e^{x-y}}$

58. $\dfrac{dy}{dx} = \dfrac{x+1}{y(y+2)}$

59. $y\cos x - \cos x + \dfrac{dy}{dx} = 0$

60. $y' = 2x\sqrt{1-y^2}$

61. $(3y^2 + 4xy)\,dx + (2xy + x^2)\,dy = 0$

62. $(x + y)\,dx - x\,dy = 0$

63. $(2y - e^x)\,dx + x\,dy = 0$

64. $(y^2 + xy)\,dx - x^2\,dy = 0$

65. $(x^2y^4 - 1)\,dx + x^3y^3\,dy = 0$

66. $y\,dx + (3x + 4y)\,dy = 0$

67. $3(y - 4x^2)\,dx + x\,dy = 0$

68. $x\,dx + (y + e^y)(x^2 + 1)\,dy = 0$

True or False? **In Exercises 69 and 70, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

69. $y' + x\sqrt{y} = x^2$ is a first-order linear differential equation.

70. $y' + xy = e^xy$ is a first-order linear differential equation.

Section Project: Weight Loss

A person's weight depends on both the number of calories consumed and the energy used. Moreover, the amount of energy used depends on a person's weight—the average amount of energy used by a person is 17.5 calories per pound per day. So, the more weight a person loses, the less energy a person uses (assuming that the person maintains a constant level of activity). An equation that can be used to model weight loss is

$$\left(\frac{dw}{dt}\right) = \frac{C}{3500} - \frac{17.5}{3500}w$$

where w is the person's weight (in pounds), t is the time in days, and C is the constant daily calorie consumption.

(a) Find the general solution of the differential equation.

(b) Consider a person who weighs 180 pounds and begins a diet of 2500 calories per day. How long will it take the person to lose 10 pounds? How long will it take the person to lose 35 pounds?

(c) Use a graphing utility to graph the solution. What is the "limiting" weight of the person?

(d) Repeat parts (b) and (c) for a person who weighs 200 pounds when the diet is started.

FOR FURTHER INFORMATION For more information on modeling weight loss, see the article "A Linear Diet Model" by Arthur C. Segal in *The College Mathematics Journal*.

Review Exercises for Chapter 6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. Determine whether the function $y = x^3$ is a solution of the differential equation $x^2y' + 3y = 6x^3$.
2. Determine whether the function $y = 2 \sin 2x$ is a solution of the differential equation $y''' - 8y = 0$.

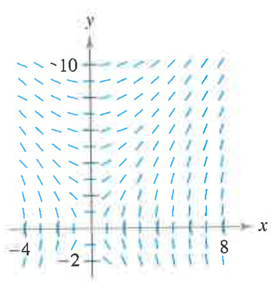
In Exercises 3–8, use integration to find a general solution of the differential equation.

3. $\dfrac{dy}{dx} = 2x^2 + 5$
4. $\dfrac{dy}{dx} = x^3 - 2x$
5. $\dfrac{dy}{dx} = \cos 2x$
6. $\dfrac{dy}{dx} = 2 \sin x$
7. $\dfrac{dy}{dx} = 2x\sqrt{x - 7}$
8. $\dfrac{dy}{dx} = 3e^{-x/3}$

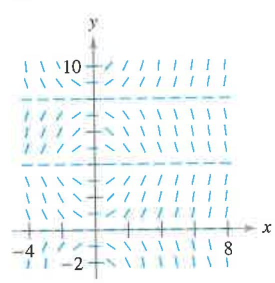
Slope Fields **In Exercises 9 and 10, a differential equation and its slope field are given. Determine the slopes (if possible) in the slope field at the points given in the table.**

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx						

9. $\dfrac{dy}{dx} = \dfrac{2x}{y}$



10. $\dfrac{dy}{dx} = x \sin\left(\dfrac{\pi y}{4}\right)$



Slope Fields **In Exercises 11–16, (a) sketch the slope field for the differential equation, and (b) use the slope field to sketch the solution that passes through the given point.**

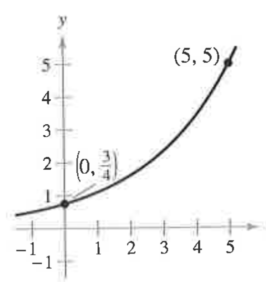
Differential Equation	Point
11. $y' = -x - 2$	$(-1, 1)$
12. $y' = 2x^2 - x$	$(0, 2)$
13. $y' = \frac{1}{4}x^2 - \frac{1}{3}x$	$(0, 3)$
14. $y' = y + 3x$	$(2, 1)$
15. $y' = \dfrac{xy}{x^2 + 4}$	$(0, 1)$
16. $y' = \dfrac{y}{x^2 + 1}$	$(0, -2)$

In Exercises 17–22, solve the differential equation.

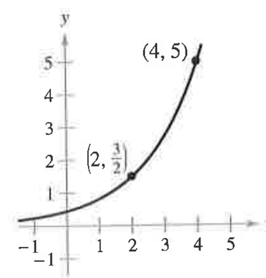
17. $\dfrac{dy}{dx} = 6 - x$
18. $\dfrac{dy}{dx} = y + 6$
19. $\dfrac{dy}{dx} = (3 + y)^2$
20. $\dfrac{dy}{dx} = 4\sqrt{y}$
21. $(2 + x)y' - xy = 0$
22. $xy' - (x + 1)y = 0$

In Exercises 23–26, find the exponential function $y = Ce^{kt}$ that passes through the two points.

23.



24.



25. $(0, 5), \left(5, \dfrac{1}{6}\right)$
26. $(1, 9), (6, 2)$

27. ***Air Pressure*** Under ideal conditions, air pressure decreases continuously with the height above sea level at a rate proportional to the pressure at that height. The barometer reads 30 inches at sea level and 15 inches at 18,000 feet. Find the barometric pressure at 35,000 feet.

28. ***Radioactive Decay*** Radioactive radium has a half-life of approximately 1599 years. The initial quantity is 5 grams. How much remains after 600 years?

29. ***Sales*** The sales S (in thousands of units) of a new product after it has been on the market for t years is given by

$$S = Ce^{k/t}.$$

(a) Find S as a function of t if 5000 units have been sold after 1 year and the saturation point for the market is 30,000 units (that is, $\lim\limits_{t\to\infty} S = 30$).

(b) How many units will have been sold after 5 years?

(c) Use a graphing utility to graph this sales function.

30. ***Sales*** The sales S (in thousands of units) of a new product after it has been on the market for t years is given by

$$S = 25(1 - e^{kt}).$$

(a) Find S as a function of t if 4000 units have been sold after 1 year.

(b) How many units will saturate this market?

(c) How many units will have been sold after 5 years?

(d) Use a graphing utility to graph this sales function.

31. ***Population Growth*** A population grows continuously at the rate of 1.5%. How long will it take the population to double?

32. ***Fuel Economy*** An automobile gets 28 miles per gallon of gasoline for speeds up to 50 miles per hour. Over 50 miles per hour, the number of miles per gallon drops at the rate of 12 percent for each 10 miles per hour.

(a) s is the speed and y is the number of miles per gallon. Find y as a function of s by solving the differential equation

$$\frac{dy}{ds} = -0.012y, \quad s > 50.$$

(b) Use the function in part (a) to complete the table.

Speed	50	55	60	65	70
Miles per Gallon					

In Exercises 33–38, solve the differential equation.

33. $\dfrac{dy}{dx} = \dfrac{x^2+3}{x}$

34. $\dfrac{dy}{dx} = \dfrac{e^{-2x}}{1+e^{-2x}}$

35. $y' - 2xy = 0$

36. $y' - e^y \sin x = 0$

37. $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{2xy}$

38. $\dfrac{dy}{dx} = \dfrac{3(x+y)}{x}$

39. Verify that the general solution $y = C_1x + C_2x^3$ satisfies the differential equation $x^2y'' - 3xy' + 3y = 0$. Then find the particular solution that satisfies the initial condition $y = 0$ and $y' = 4$ when $x = 2$.

40. ***Vertical Motion*** A falling object encounters air resistance that is proportional to its velocity. The acceleration due to gravity is -9.8 meters per second per second. The net change in velocity is $dv/dt = kv - 9.8$.

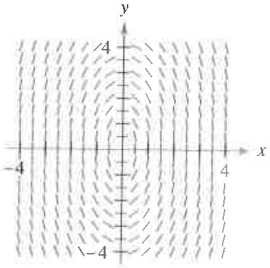
(a) Find the velocity of the object as a function of time if the initial velocity is v_0.

(b) Use the result of part (a) to find the limit of the velocity as t approaches infinity.

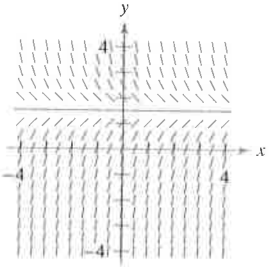
(c) Integrate the velocity function found in part (a) to find the position function s.

Slope Fields **In Exercises 41 and 42, sketch a few solutions of the differential equation on the slope field and then find the general solution analytically. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

41. $\dfrac{dy}{dx} = -\dfrac{4x}{y}$



42. $\dfrac{dy}{dx} = 3 - 2y$



In Exercises 43 and 44, the logistic equation models the growth of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution $P(t)$.

43. $P(t) = \dfrac{7200}{1 + 44e^{-0.55t}}$

44. $P(t) = \dfrac{4800}{1 + 14e^{-0.15t}}$

45. ***Environment*** A conservation department releases 1200 brook trout into a lake. It is estimated that the carrying capacity of the lake for the species is 20,400. After the first year, there are 2000 brook trout in the lake.

(a) Write a logistic equation that models the number of brook trout in the lake.

(b) Find the number of brook trout in the lake after 8 years.

(c) When will the number of brook trout reach 10,000?

46. ***Environment*** Write a logistic differential equation that models the growth rate of the brook trout population in Exercise 45. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answers.

In Exercises 47–56, solve the first-order linear differential equation.

47. $y' - y = 8$

48. $e^x y' + 4e^x y = 1$

49. $4y' = e^{x/4} + y$

50. $\dfrac{dy}{dx} - \dfrac{5y}{x^2} = \dfrac{1}{x^2}$

51. $(x-2)y' + y = 1$

52. $(x+3)y' + 2y = 2(x+3)^2$

53. $(3y + \sin 2x)\,dx - dy = 0$

54. $dy = (y \tan x + 2e^x)\,dx$

55. $y' + 5y = e^{5x}$

56. $xy' - ay = bx^4$

In Exercises 57–60, solve the Bernoulli differential equation.

57. $y' + y = xy^2$ $\left[\textit{Hint: } \int xe^{-x}\,dx = (-x-1)e^{-x}\right]$

58. $y' + 2xy = xy^2$

59. $y' + \left(\dfrac{1}{x}\right)y = \dfrac{y^3}{x^2}$

60. $xy' + y = xy^2$

In Exercises 61–64, write an example of the given differential equation. Then solve your equation.

61. Homogeneous differential equation

62. Logistic differential equation

63. First-order linear differential equation

64. Bernoulli differential equation

P.S. Problem Solving

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. The differential equation

$$\frac{dy}{dt} = ky^{1+\varepsilon}$$

where k and ε are positive constants, is called the **doomsday equation.**

(a) Solve the doomsday equation

$$\frac{dy}{dt} = y^{1.01}$$

given that $y(0) = 1$. Find the time T at which

$$\lim_{t \to T^-} y(t) = \infty.$$

(b) Solve the doomsday equation

$$\frac{dy}{dt} = ky^{1+\varepsilon}$$

given that $y(0) = y_0$. Explain why this equation is called the doomsday equation.

2. A thermometer is taken from a room at 72°F to the outdoors, where the temperature is 20°F. The reading drops to 48°F after 1 minute. Determine the reading on the thermometer after 5 minutes.

3. Let S represent sales of a new product (in thousands of units), let L represent the maximum level of sales (in thousands of units), and let t represent time (in months). The rate of change of S with respect to t varies jointly as the product of S and $L - S$.

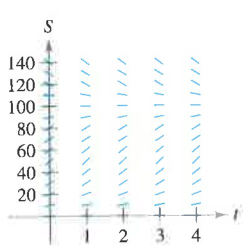
(a) Write the differential equation for the sales model if $L = 100$, $S = 10$ when $t = 0$, and $S = 20$ when $t = 1$. Verify that

$$S = \frac{L}{1 + Ce^{-kt}}.$$

(b) At what time is the growth in sales increasing most rapidly?

(c) Use a graphing utility to graph the sales function.

(d) Sketch the solution from part (a) on the slope field shown in the figure below. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.



(e) If the estimated maximum level of sales is correct, use the slope field to describe the shape of the solution curves for sales if, at some period of time, sales exceed L.

4. Another model that can be used to represent population growth is the **Gompertz equation,** which is the solution of the differential equation

$$\frac{dy}{dt} = k \ln\left(\frac{L}{y}\right)y$$

where k is a constant and L is the carrying capacity.

(a) Solve the differential equation.

(b) Use a graphing utility to graph the slope field for the differential equation when $k = 0.05$ and $L = 1000$.

(c) Describe the behavior of the graph as $t \to \infty$.

(d) Graph the equation you found in part (a) for $L = 5000$, $y_0 = 500$, and $k = 0.02$. Determine the concavity of the graph and how it compares with the general solution of the logistical differential equation.

5. Show that the logistic equation

$$y = \frac{L}{1 + be^{-kt}}$$

can be written as

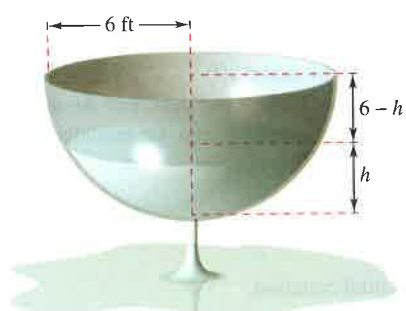
$$y = \frac{1}{2}L\left[1 + \tanh\left(\frac{1}{2}k\left(t - \frac{\ln b}{k}\right)\right)\right].$$

What can you conclude about the graph of the logistic equation?

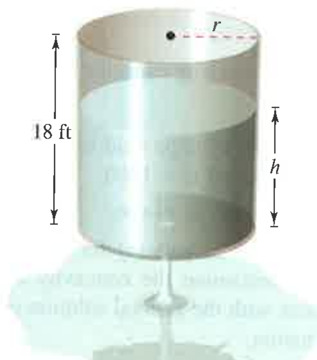
6. **Torricelli's Law** states that water will flow from an opening at the bottom of a tank with the same speed that it would attain falling from the surface of the water to the opening. One of the forms of Torricelli's Law is

$$A(h)\frac{dh}{dt} = -k\sqrt{2gh}$$

where h is the height of the water in the tank, k is the area of the opening at the bottom of the tank, $A(h)$ is the horizontal cross-sectional area at height h, and g is the acceleration due to gravity ($g \approx 32$ feet per second per second). A hemispherical water tank has a radius of 6 feet. When the tank is full, a circular valve with a radius of 1 inch is opened at the bottom, as shown in the figure. How long will it take for the tank to drain completely?



7. The cylindrical water tank shown in the figure has a height of 18 feet. When the tank is full, a circular valve is opened at the bottom of the tank. After 30 minutes, the depth of the water is 12 feet.



(a) How long will it take for the tank to drain completely?

(b) What is the depth of the water in the tank after 1 hour?

8. Suppose the tank in Exercise 7 has a height of 20 feet, a radius of 8 feet, and the valve is circular with a radius of 2 inches. The tank is full when the valve is opened. How long will it take for the tank to drain completely?

9. In hilly areas, radio reception may be poor. Consider a situation where an FM transmitter is located at the point $(-1, 1)$ behind a hill modeled by the graph of

$$y = x - x^2$$

and a radio receiver is on the opposite side of the hill. (Assume that the x-axis represents ground level at the base of the hill.)

(a) What is the closest position $(x, 0)$ the radio can be to the hill so that reception is unobstructed?

(b) Write the closest position $(x, 0)$ of the radio with x represented as a function of h if the transmitter is located at $(-1, h)$.

(c) Use a graphing utility to graph the function for x in part (b). Determine the vertical asymptote of the function and interpret the result.

10. Biomass is a measure of an amount of living matter in an ecosystem. Suppose the biomass $s(t)$ in a given ecosystem increases at a rate of about 3.5 tons per year, and decreases by about 1.9% per year. This situation can be modeled by the differential equation

$$\frac{ds}{dt} = 3.5 - 0.019s.$$

(a) Solve the differential equation.

(b) Use a graphing utility to graph the slope field for the differential equation. What do you notice?

(c) Explain what happens as $t \to \infty$.

In Exercises 11–13, a medical researcher wants to determine the concentration C (in moles per liter) of a tracer drug injected into a moving fluid. Solve this problem by considering a single-compartment dilution model (see figure). Assume that the fluid is continuously mixed and that the volume of the fluid in the compartment is constant.

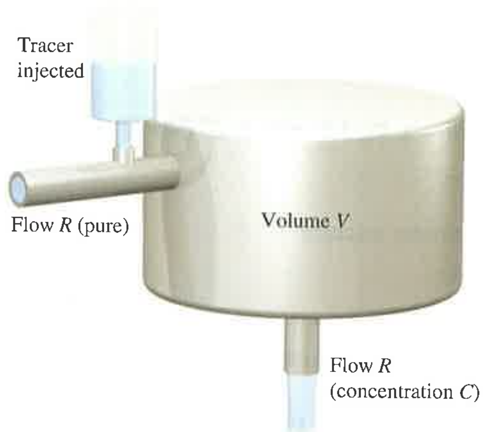


Figure for 11–13

11. If the tracer is injected instantaneously at time $t = 0$, then the concentration of the fluid in the compartment begins diluting according to the differential equation

$$\frac{dC}{dt} = \left(-\frac{R}{V}\right)C, \quad C = C_0 \text{ when } t = 0.$$

(a) Solve this differential equation to find the concentration C as a function of time t.

(b) Find the limit of C as $t \to \infty$.

12. Use the solution of the differential equation in Exercise 11 to find the concentration C as a function of time t, and use a graphing utility to graph the function.

(a) $V = 2$ liters, $R = 0.5$ liter per minute, and $C_0 = 0.6$ mole per liter

(b) $V = 2$ liters, $R = 1.5$ liters per minute, and $C_0 = 0.6$ mole per liter

13. In Exercises 11 and 12, it was assumed that there was a single initial injection of the tracer drug into the compartment. Now consider the case in which the tracer is continuously injected (beginning at $t = 0$) at the rate of Q moles per minute. Considering Q to be negligible compared with R, use the differential equation

$$\frac{dC}{dt} = \frac{Q}{V} - \left(\frac{R}{V}\right)C, \quad C = 0 \text{ when } t = 0.$$

(a) Solve this differential equation to find the concentration C as a function of time t.

(b) Find the limit of C as $t \to \infty$.