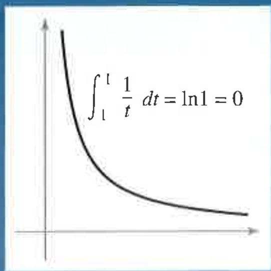
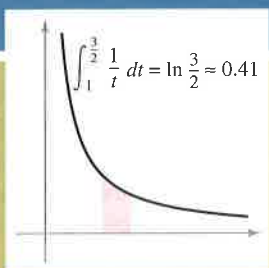
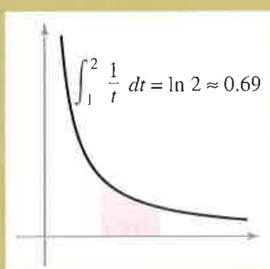
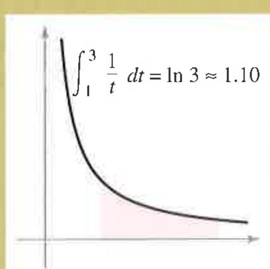
5 Logarithmic, Exponential, and Other Transcendental Functions









In Chapter 5, you will see how the function $f(x) = 1/x$ can be used to define the natural logarithmic function. To do this, consider the definite integral

$$\int_1^x \frac{1}{t}\,dt.$$

When $x < 1$, the value of this definite integral is negative. When $x = 1$, the value is 0. When $x > 1$, the value is positive.

A geyser is a hot spring that erupts periodically when groundwater in a confined space boils and produces steam. The steam forces overlying water up and out through an opening on Earth's surface. The temperature at which water boils is affected by pressure. Do you think an *increase* or a *decrease* in pressure causes water to boil at a lower temperature? Why?

Brian Maslyar/Index Stock

Section 5.1 The Natural Logarithmic Function: Differentiation

- **Develop and use properties of the natural logarithmic function.**
- **Understand the definition of the number *e*.**
- **Find derivatives of functions involving the natural logarithmic function.**

JOHN NAPIER (1550–1617)

Logarithms were invented by the Scottish mathematician John Napier. Although he did not introduce the *natural* logarithmic function, it is sometimes called the *Napierian* logarithm.

The Natural Logarithmic Function

Recall that the General Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1 \qquad \text{General Power Rule}$$

has an important disclaimer—it doesn't apply when $n = -1$. Consequently, you have not yet found an antiderivative for the function $f(x) = 1/x$. In this section, you will use the Second Fundamental Theorem of Calculus to *define* such a function. This antiderivative is a function that you have not encountered previously in the text. It is neither algebraic nor trigonometric, but falls into a new class of functions called *logarithmic functions*. This particular function is the **natural logarithmic function.**

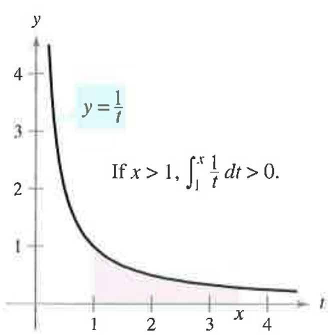
Definition of the Natural Logarithmic Function

The **natural logarithmic function** is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

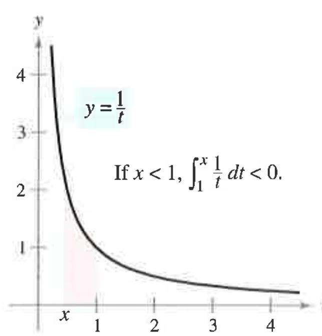
The domain of the natural logarithmic function is the set of all positive real numbers.

From the definition, you can see that $\ln x$ is positive for $x > 1$ and negative for $0 < x < 1$, as shown in Figure 5.1. Moreover, $\ln(1) = 0$, because the upper and lower limits of integration are equal when $x = 1$.



If $x > 1$, then $\ln x > 0$.

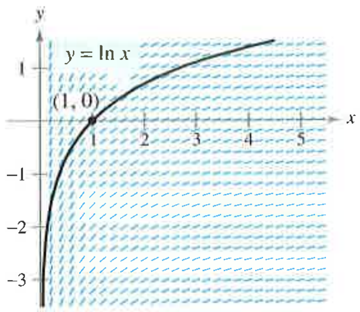
Figure 5.1



If $0 < x < 1$, then $\ln x < 0$.

EXPLORATION

Graphing the Natural Logarithmic Function Using *only* the definition of the natural logarithmic function, sketch a graph of the function. Explain your reasoning.



Each small line segment has a slope of $\frac{1}{x}$.
Figure 5.2

NOTE Slope fields can be helpful in getting a visual perspective of the directions of the solutions of a differential equation.

To sketch the graph of $y = \ln x$, you can think of the natural logarithmic function as an *antiderivative* given by the differential equation

$$\frac{dy}{dx} = \frac{1}{x}.$$

Figure 5.2 is a computer-generated graph, called a *slope (or direction) field*, showing small line segments of slope $1/x$. The graph of $y = \ln x$ is the solution that passes through the point $(1, 0)$. You will study slope fields in Section 6.1.

The following theorem lists some basic properties of the natural logarithmic function.

THEOREM 5.1 Properties of the Natural Logarithmic Function

The natural logarithmic function has the following properties.

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

Proof The domain of $f(x) = \ln x$ is $(0, \infty)$ by definition. Moreover, the function is continuous because it is differentiable. It is increasing because its derivative

$$f'(x) = \frac{1}{x} \qquad \text{First derivative}$$

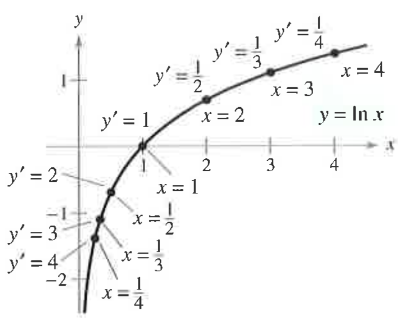
is positive for $x > 0$, as shown in Figure 5.3. It is concave downward because

$$f''(x) = -\frac{1}{x^2} \qquad \text{Second derivative}$$

is negative for $x > 0$. The proof that f is one-to-one is left as an exercise (see Exercise 111). The following limits imply that its range is the entire real line.

$$\lim_{x\to 0^+} \ln x = -\infty \quad \text{and} \quad \lim_{x\to\infty} \ln x = \infty$$

Verification of these two limits is given in Appendix A.



The natural logarithmic function is increasing, and its graph is concave downward.
Figure 5.3

Using the definition of the natural logarithmic function, you can prove several important properties involving operations with natural logarithms. If you are already familiar with logarithms, you will recognize that these properties are characteristic of all logarithms.

THEOREM 5.2 Logarithmic Properties

If a and b are positive numbers and n is rational, then the following properties are true.

1. $\ln(1) = 0$
2. $\ln(ab) = \ln a + \ln b$
3. $\ln(a^n) = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

LOGARITHMS

Napier coined the term *logarithm*, from the two Greek words *logos* (or ratio) and *arithmos* (or number), to describe the theory that he spent 20 years developing and that first appeared in the book *Mirifici Logarithmorum canonis descriptio* (A Description of the Marvelous Rule of Logarithms).

Proof The first property has already been discussed. The proof of the second property follows from the fact that two antiderivatives of the same function differ at most by a constant. From the Second Fundamental Theorem of Calculus and the definition of the natural logarithmic function, you know that

$$\frac{d}{dx}[\ln x] = \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right] = \frac{1}{x}.$$

So, consider the two derivatives

$$\frac{d}{dx}[\ln(ax)] = \frac{a}{ax} = \frac{1}{x}$$

and

$$\frac{d}{dx}[\ln a + \ln x] = 0 + \frac{1}{x} = \frac{1}{x}.$$

Because $\ln(ax)$ and $(\ln a + \ln x)$ are both antiderivatives of $1/x$, they must differ at most by a constant.

$$\ln(ax) = \ln a + \ln x + C$$

By letting $x = 1$, you can see that $C = 0$. The third property can be proved similarly by comparing the derivatives of $\ln(x^n)$ and $n \ln x$. Finally, using the second and third properties, you can prove the fourth property.

$$\ln\left(\frac{a}{b}\right) = \ln[a(b^{-1})] = \ln a + \ln(b^{-1}) = \ln a - \ln b$$

Example 1 shows how logarithmic properties can be used to expand logarithmic expressions.

EXAMPLE 1 Expanding Logarithmic Expressions

a. $\ln\dfrac{10}{9} = \ln 10 - \ln 9$ Property 4

b. $\ln\sqrt{3x+2} = \ln(3x+2)^{1/2}$ Rewrite with rational exponent.

$= \dfrac{1}{2}\ln(3x+2)$ Property 3

c. $\ln\dfrac{6x}{5} = \ln(6x) - \ln 5$ Property 4

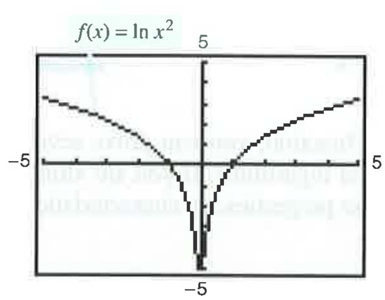
$= \ln 6 + \ln x - \ln 5$ Property 2

d. $\ln\dfrac{(x^2+3)^2}{x\sqrt[3]{x^2+1}} = \ln(x^2+3)^2 - \ln\left(x\sqrt[3]{x^2+1}\right)$

$= 2\ln(x^2+3) - [\ln x + \ln(x^2+1)^{1/3}]$

$= 2\ln(x^2+3) - \ln x - \ln(x^2+1)^{1/3}$

$= 2\ln(x^2+3) - \ln x - \dfrac{1}{3}\ln(x^2+1)$



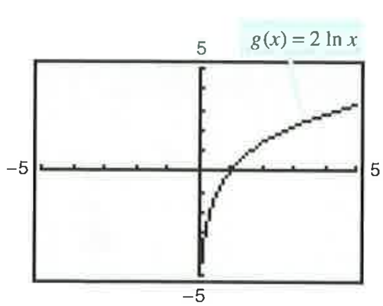
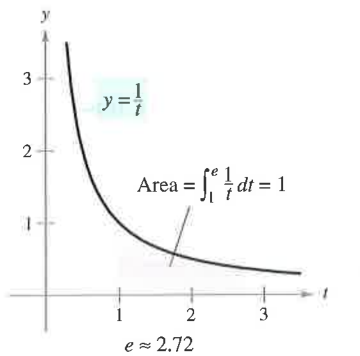


Figure 5.4

When using the properties of logarithms to rewrite logarithmic functions, you must check to see whether the domain of the rewritten function is the same as the domain of the original. For instance, the domain of $f(x) = \ln x^2$ is all real numbers except $x = 0$, and the domain of $g(x) = 2\ln x$ is all positive real numbers. (See Figure 5.4.)



e is the base for the natural logarithm because $\ln e = 1$.
Figure 5.5

The Number e

It is likely that you have studied logarithms in an algebra course. There, without the benefit of calculus, logarithms would have been defined in terms of a **base** number. For example, common logarithms have a base of 10 and therefore $\log_{10} 10 = 1$. (You will learn more about this in Section 5.5.)

The **base for the natural logarithm** is defined using the fact that the natural logarithmic function is continuous, is one-to-one, and has a range of $(-\infty, \infty)$. So, there must be a unique real number x such that $\ln x = 1$, as shown in Figure 5.5. This number is denoted by the letter e. It can be shown that e is irrational and has the following decimal approximation.

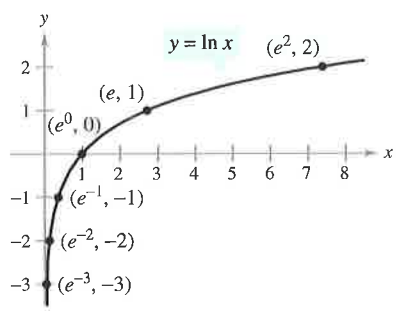
$$e \approx 2.71828182846$$

Definition of e

The letter e denotes the positive real number such that

$$\ln e = \int_1^e \frac{1}{t}\,dt = 1.$$

FOR FURTHER INFORMATION To learn more about the number e, see the article "Unexpected Occurrences of the Number e" by Harris S. Shultz and Bill Leonard in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.



If $x = e^n$, then $\ln x = n$.
Figure 5.6

Once you know that $\ln e = 1$, you can use logarithmic properties to evaluate the natural logarithms of several other numbers. For example, by using the property

$$\begin{aligned}\ln(e^n) &= n \ln e\\ &= n(1)\\ &= n\end{aligned}$$

you can evaluate $\ln(e^n)$ for various values of n, as shown in the table and in Figure 5.6.

x	$\frac{1}{e^3} \approx 0.050$	$\frac{1}{e^2} \approx 0.135$	$\frac{1}{e} \approx 0.368$	$e^0 = 1$	$e \approx 2.718$	$e^2 \approx 7.389$
$\ln x$	-3	-2	-1	0	1	2

The logarithms shown in the table above are convenient because the x-values are integer powers of e. Most logarithmic expressions are, however, best evaluated with a calculator.

EXAMPLE 2 Evaluating Natural Logarithmic Expressions

a. $\ln 2 \approx 0.693$

b. $\ln 32 \approx 3.466$

c. $\ln 0.1 \approx -2.303$

The Derivative of the Natural Logarithmic Function

The derivative of the natural logarithmic function is given in Theorem 5.3. The first part of the theorem follows from the definition of the natural logarithmic function as an antiderivative. The second part of the theorem is simply the Chain Rule version of the first part.

THEOREM 5.3 Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}, \quad x > 0$ **2.** $\dfrac{d}{dx}[\ln u] = \dfrac{1}{u}\dfrac{du}{dx} = \dfrac{u'}{u}, \quad u > 0$

EXAMPLE 3 Differentiation of Logarithmic Functions

a. $\dfrac{d}{dx}[\ln(2x)] = \dfrac{u'}{u} = \dfrac{2}{2x} = \dfrac{1}{x}$ $\quad u = 2x$

b. $\dfrac{d}{dx}[\ln(x^2+1)] = \dfrac{u'}{u} = \dfrac{2x}{x^2+1}$ $\quad u = x^2 + 1$

c. $\dfrac{d}{dx}[x\ln x] = x\left(\dfrac{d}{dx}[\ln x]\right) + (\ln x)\left(\dfrac{d}{dx}[x]\right)$ $\quad$ Product Rule

$$= x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$$

d. $\dfrac{d}{dx}[(\ln x)^3] = 3(\ln x)^2\dfrac{d}{dx}[\ln x]$ $\quad$ Chain Rule

$$= 3(\ln x)^2\frac{1}{x}$$

EXPLORATION

Use a graphing utility to graph

$$y_1 = \frac{1}{x}$$

and

$$y_2 = \frac{d}{dx}[\ln x]$$

in the same viewing window, in which $0.1 \le x \le 5$ and $-2 \le y \le 8$. Explain why the graphs appear to be identical.

Napier used logarithmic properties to simplify *calculations* involving products, quotients, and powers. Of course, given the availability of calculators, there is now little need for this particular application of logarithms. However, there is great value in using logarithmic properties to simplify *differentiation* involving products, quotients, and powers.

EXAMPLE 4 Logarithmic Properties as Aids to Differentiation

Differentiate $f(x) = \ln\sqrt{x+1}$.

Solution Because

$$f(x) = \ln\sqrt{x+1} = \ln(x+1)^{1/2} = \frac{1}{2}\ln(x+1) \qquad \text{Rewrite before differentiating.}$$

you can write

$$f'(x) = \frac{1}{2}\left(\frac{1}{x+1}\right) = \frac{1}{2(x+1)}. \qquad \text{Differentiate.}$$

indicates that in the HM mathSpace® *CD-ROM and the online* Eduspace® *system for this text, you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

EXAMPLE 5 Logarithmic Properties as Aids to Differentiation

Differentiate $f(x) = \ln \dfrac{x(x^2+1)^2}{\sqrt{2x^3-1}}$.

Solution

$$f(x) = \ln \frac{x(x^2+1)^2}{\sqrt{2x^3-1}} \qquad \text{Write original function.}$$

$$= \ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1) \qquad \text{Rewrite before differentiating.}$$

$$f'(x) = \frac{1}{x} + 2\left(\frac{2x}{x^2+1}\right) - \frac{1}{2}\left(\frac{6x^2}{2x^3-1}\right) \qquad \text{Differentiate.}$$

$$= \frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1} \qquad \text{Simplify.}$$

NOTE In Examples 4 and 5, be sure you see the benefit of applying logarithmic properties *before* differentiating. Consider, for instance, the difficulty of direct differentiation of the function given in Example 5.

On occasion, it is convenient to use logarithms as aids in differentiating *nonlogarithmic* functions. This procedure is called **logarithmic differentiation.**

EXAMPLE 6 Logarithmic Differentiation

Find the derivative of

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2.$$

Solution Note that $y > 0$ for all $x \neq 2$. So, $\ln y$ is defined. Begin by taking the natural logarithm of each side of the equation. Then apply logarithmic properties and differentiate implicitly. Finally, solve for y'.

$$y = \frac{(x-2)^2}{\sqrt{x^2+1}}, \quad x \neq 2 \qquad \text{Write original equation.}$$

$$\ln y = \ln \frac{(x-2)^2}{\sqrt{x^2+1}} \qquad \text{Take natural log of each side.}$$

$$\ln y = 2\ln(x-2) - \frac{1}{2}\ln(x^2+1) \qquad \text{Logarithmic properties}$$

$$\frac{y'}{y} = 2\left(\frac{1}{x-2}\right) - \frac{1}{2}\left(\frac{2x}{x^2+1}\right) \qquad \text{Differentiate.}$$

$$= \frac{2}{x-2} - \frac{x}{x^2+1} \qquad \text{Simplify.}$$

$$y' = y\left(\frac{2}{x-2} - \frac{x}{x^2+1}\right) \qquad \text{Solve for } y'.$$

$$= \frac{(x-2)^2}{\sqrt{x^2+1}}\left[\frac{x^2+2x+2}{(x-2)(x^2+1)}\right] \qquad \text{Substitute for } y.$$

$$= \frac{(x-2)(x^2+2x+2)}{(x^2+1)^{3/2}} \qquad \text{Simplify.}$$

Because the natural logarithm is undefined for negative numbers, you will often encounter expressions of the form $\ln|u|$. The following theorem states that you can differentiate functions of the form $y = \ln|u|$ as if the absolute value sign were not present.

THEOREM 5.4 Derivative Involving Absolute Value

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}.$$

Proof If $u > 0$, then $|u| = u$, and the result follows from Theorem 5.3. If $u < 0$, then $|u| = -u$, and you have

$$\begin{aligned}\frac{d}{dx}[\ln|u|] &= \frac{d}{dx}[\ln(-u)]\\ &= \frac{-u'}{-u}\\ &= \frac{u'}{u}.\end{aligned}$$

EXAMPLE 7 Derivative Involving Absolute Value

Find the derivative of

$$f(x) = \ln|\cos x|.$$

Solution Using Theorem 5.4, let $u = \cos x$ and write

$$\begin{aligned}\frac{d}{dx}[\ln|\cos x|] &= \frac{u'}{u} && \frac{d}{dx}[\ln|u|] = \frac{u'}{u}\\ &= \frac{-\sin x}{\cos x} && u = \cos x\\ &= -\tan x. && \text{Simplify.}\end{aligned}$$

EXAMPLE 8 Finding Relative Extrema

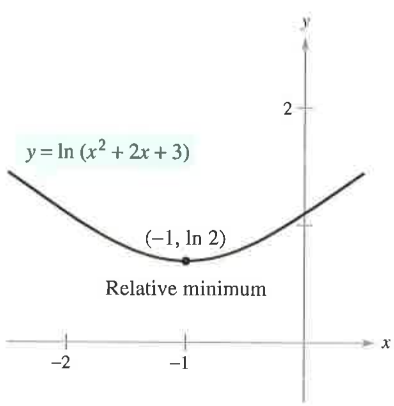
Locate the relative extrema of

$$y = \ln(x^2 + 2x + 3).$$

Solution Differentiating y, you obtain

$$\frac{dy}{dx} = \frac{2x + 2}{x^2 + 2x + 3}.$$

Because $dy/dx = 0$ when $x = -1$, you can apply the First Derivative Test and conclude that the point $(-1, \ln 2)$ is a relative minimum. Because there are no other critical points, it follows that this is the only relative extremum (see Figure 5.7).



The derivative of y changes from negative to positive at $x = -1$.
Figure 5.7

Exercises for Section 5.1

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. Complete the table below. Use a graphing utility and Simpson's Rule with $n = 10$ to approximate the integral $\int_1^x (1/t)\,dt$.

x	0.5	1.5	2	2.5	3	3.5	4
$\int_1^x (1/t)\,dt$							

2. (a) Plot the points generated in Exercise 1 and connect them with a smooth curve. Compare the result with the graph of $y = \ln x$.

(b) Use a graphing utility to graph $y = \int_1^x (1/t)\,dt$ for $0.2 \le x \le 4$. Compare the result with the graph of $y = \ln x$.

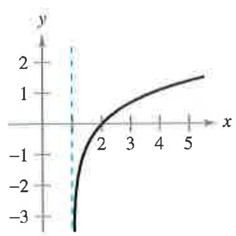
In Exercises 3–6, use a graphing utility to evaluate the logarithm by (a) using the natural logarithm key and (b) using the integration capabilities to evaluate the integral $\int_1^x (1/t)\,dt$.

3. $\ln 45$ **4.** $\ln 8.3$

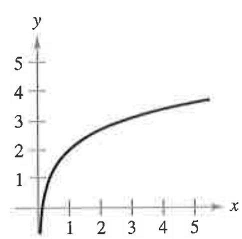
5. $\ln 0.8$ **6.** $\ln 0.6$

In Exercises 7–10, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]

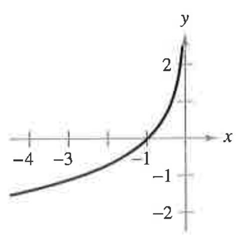
(a)



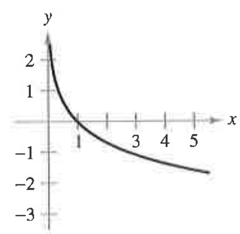
(b)



(c)



(d)



7. $f(x) = \ln x + 2$ **8.** $f(x) = -\ln x$

9. $f(x) = \ln(x - 1)$ **10.** $f(x) = -\ln(-x)$

In Exercises 11–16, sketch the graph of the function and state its domain.

11. $f(x) = 3 \ln x$ **12.** $f(x) = -2 \ln x$

13. $f(x) = \ln 2x$ **14.** $f(x) = \ln|x|$

15. $f(x) = \ln(x - 1)$ **16.** $g(x) = 2 + \ln x$

In Exercises 17 and 18, use the properties of logarithms to approximate the indicated logarithms, given that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$.

17. (a) $\ln 6$ (b) $\ln \frac{2}{3}$ (c) $\ln 81$ (d) $\ln \sqrt{3}$

18. (a) $\ln 0.25$ (b) $\ln 24$ (c) $\ln \sqrt[3]{12}$ (d) $\ln \frac{1}{72}$

In Exercises 19–28, use the properties of logarithms to expand the logarithmic expression.

19. $\ln \frac{2}{3}$ **20.** $\ln \sqrt{2^3}$

21. $\ln \dfrac{xy}{z}$ **22.** $\ln(xyz)$

23. $\ln \sqrt[3]{a^2 + 1}$ **24.** $\ln \sqrt{a - 1}$

25. $\ln\left(\dfrac{x^2 - 1}{x^3}\right)^3$ **26.** $\ln(3e^2)$

27. $\ln z(z - 1)^2$ **28.** $\ln \dfrac{1}{e}$

In Exercises 29–34, write the expression as a logarithm of a single quantity.

29. $\ln(x - 2) - \ln(x + 2)$ **30.** $3 \ln x + 2 \ln y - 4 \ln z$

31. $\frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)]$

32. $2[\ln x - \ln(x + 1) - \ln(x - 1)]$

33. $2 \ln 3 - \frac{1}{2}\ln(x^2 + 1)$

34. $\frac{3}{2}[\ln(x^2 + 1) - \ln(x + 1) - \ln(x - 1)]$

In Exercises 35 and 36, (a) verify that $f = g$ by using a graphing utility to graph f and g in the same viewing window. (b) Then verify that $f = g$ algebraically.

35. $f(x) = \ln \dfrac{x^2}{4}$, $x > 0$, $g(x) = 2 \ln x - \ln 4$

36. $f(x) = \ln\sqrt{x(x^2 + 1)}$, $g(x) = \frac{1}{2}[\ln x + \ln(x^2 + 1)]$

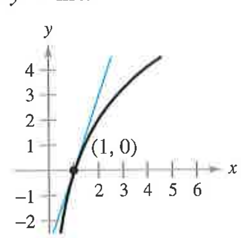
In Exercises 37–40, find the limit.

37. $\lim\limits_{x \to 3^+} \ln(x - 3)$ **38.** $\lim\limits_{x \to 6^-} \ln(6 - x)$

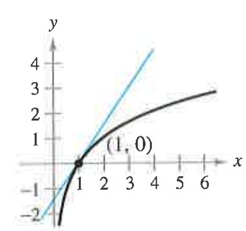
39. $\lim\limits_{x \to 2^-} \ln[x^2(3 - x)]$ **40.** $\lim\limits_{x \to 5^+} \ln \dfrac{x}{\sqrt{x - 4}}$

In Exercises 41–44, find an equation of the tangent line to the graph of the logarithmic function at the point (1, 0).

41. $y = \ln x^3$



42. $y = \ln x^{3/2}$



43. $y = \ln x^2$

44. $y = \ln x^{1/2}$

In Exercises 45–70, find the derivative of the function.

45. $g(x) = \ln x^2$

46. $h(x) = \ln(2x^2 + 1)$

47. $y = (\ln x)^4$

48. $y = x \ln x$

49. $y = \ln\left(x\sqrt{x^2 - 1}\right)$

50. $y = \ln\sqrt{x^2 - 4}$

51. $f(x) = \ln\left(\dfrac{x}{x^2 + 1}\right)$

52. $f(x) = \ln\left(\dfrac{2x}{x + 3}\right)$

53. $g(t) = \dfrac{\ln t}{t^2}$

54. $h(t) = \dfrac{\ln t}{t}$

55. $y = \ln(\ln x^2)$

56. $y = \ln(\ln x)$

57. $y = \ln\sqrt{\dfrac{x + 1}{x - 1}}$

58. $y = \ln\sqrt[3]{\dfrac{x - 1}{x + 1}}$

59. $f(x) = \ln\left(\dfrac{\sqrt{4 + x^2}}{x}\right)$

60. $f(x) = \ln\left(x + \sqrt{4 + x^2}\right)$

61. $y = \dfrac{-\sqrt{x^2 + 1}}{x} + \ln\left(x + \sqrt{x^2 + 1}\right)$

62. $y = \dfrac{-\sqrt{x^2 + 4}}{2x^2} - \dfrac{1}{4}\ln\left(\dfrac{2 + \sqrt{x^2 + 4}}{x}\right)$

63. $y = \ln|\sin x|$

64. $y = \ln|\csc x|$

65. $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

66. $y = \ln|\sec x + \tan x|$

67. $y = \ln\left|\dfrac{-1 + \sin x}{2 + \sin x}\right|$

68. $y = \ln\sqrt{2 + \cos^2 x}$

69. $f(x) = \displaystyle\int_2^{\ln(2x)} (t + 1)\, dt$

70. $g(x) = \displaystyle\int_1^{\ln x} (t^2 + 3)\, dt$

In Exercises 71–76, (a) find an equation of the tangent line to the graph of f at the given point, (b) use a graphing utility to graph the function and its tangent line at the point, and (c) use the *derivative* feature of a graphing utility to confirm your results.

71. $f(x) = 3x^2 - \ln x, \quad (1, 3)$

72. $f(x) = 4 - x^2 - \ln\left(\frac{1}{2}x + 1\right), \quad (0, 4)$

73. $f(x) = \ln\sqrt{1 + \sin^2 x}, \quad \left(\dfrac{\pi}{4}, \ln\sqrt{\dfrac{3}{2}}\right)$

74. $f(x) = \sin(2x)\ln(x^2), \quad (1, 0)$

75. $f(x) = x^3 \ln x, \quad (1, 0)$

76. $f(x) = \dfrac{1}{2}x\ln(x^2), \quad (-1, 0)$

In Exercises 77 and 78, use implicit differentiation to find dy/dx.

77. $x^2 - 3\ln y + y^2 = 10$

78. $\ln xy + 5x = 30$

In Exercises 79 and 80, use implicit differentiation to find an equation of the tangent line to the graph at the given point.

79. $x + y - 1 = \ln(x^2 + y^2), \quad (1, 0)$

80. $y^2 + \ln(xy) = 2, \quad (e, 1)$

In Exercises 81 and 82, show that the function is a solution of the differential equation.

	Function	*Differential Equation*
81.	$y = 2\ln x + 3$	$xy'' + y' = 0$
82.	$y = x\ln x - 4x$	$x + y - xy' = 0$

In Exercises 83–88, locate any relative extrema and inflection points. Use a graphing utility to confirm your results.

83. $y = \dfrac{x^2}{2} - \ln x$

84. $y = x - \ln x$

85. $y = x\ln x$

86. $y = \dfrac{\ln x}{x}$

87. $y = \dfrac{x}{\ln x}$

88. $y = x^2 \ln \dfrac{x}{4}$

Linear and Quadratic Approximations **In Exercises 89 and 90, use a graphing utility to graph the function. Then graph**

$$P_1(x) = f(1) + f'(1)(x - 1)$$

and

$$P_2(x) = f(1) + f'(1)(x - 1) + \tfrac{1}{2}f''(1)(x - 1)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = 1$.

89. $f(x) = \ln x$

90. $f(x) = x\ln x$

In Exercises 91 and 92, use Newton's Method to approximate, to three decimal places, the x-coordinate of the point of intersection of the graphs of the two equations. Use a graphing utility to verify your result.

91. $y = \ln x, \quad y = -x$

92. $y = \ln x, \quad y = 3 - x$

In Exercises 93–98, use logarithmic differentiation to find dy/dx.

93. $y = x\sqrt{x^2 - 1}$

94. $y = \sqrt{(x - 1)(x - 2)(x - 3)}$

95. $y = \dfrac{x^2\sqrt{3x - 2}}{(x - 1)^2}$

96. $y = \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$

97. $y = \dfrac{x(x - 1)^{3/2}}{\sqrt{x + 1}}$

98. $y = \dfrac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

Writing About Concepts

99. In your own words, state the properties of the natural logarithmic function.

100. Define the base for the natural logarithmic function.

101. Let f be a function that is positive and differentiable on the entire real line. Let $g(x) = \ln f(x)$.

(a) If g is increasing, must f be increasing? Explain.

(b) If the graph of f is concave upward, must the graph of g be concave upward? Explain.

Writing About Concepts *(continued)*

102. Consider the function $f(x) = x - 2 \ln x$ on $[1, 3]$.

(a) Explain why Rolle's Theorem (Section 3.2) does not apply.

(b) Do you think the conclusion of Rolle's Theorem is true for f? Explain.

True or False? **In Exercises 103 and 104, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

103. $\ln(x + 25) = \ln x + \ln 25$

104. If $y = \ln \pi$, then $y' = 1/\pi$.

105. ***Home Mortgage*** The term t (in years) of a \$120,000 home mortgage at 10% interest can be approximated by

$$t = \frac{5.315}{-6.7968 + \ln x}, \quad x > 1000$$

where x is the monthly payment in dollars.

(a) Use a graphing utility to graph the model.

(b) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1167.41. What is the total amount paid?

(c) Use the model to approximate the term of a home mortgage for which the monthly payment is \$1068.45. What is the total amount paid?

(d) Find the instantaneous rate of change of t with respect to x when $x = 1167.41$ and $x = 1068.45$.

(e) Write a short paragraph describing the benefit of the higher monthly payment.

106. ***Sound Intensity*** The relationship between the number of decibels β and the intensity of a sound I in watts per centimeter squared is

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-16}}\right).$$

Use the properties of logarithms to write the formula in simpler form, and determine the number of decibels of a sound with an intensity of 10^{-10} watts per square centimeter.

107. ***Modeling Data*** The table shows the temperature T (°F) at which water boils at selected pressures p (pounds per square inch). *(Source: Standard Handbook of Mechanical Engineers)*

p	5	10	14.696 (1 atm)	20
T	162.24°	193.21°	212.00°	227.96°

p	30	40	60	80	100
T	250.33°	267.25°	292.71°	312.03°	327.81°

A model that approximates the data is

$$T = 87.97 + 34.96 \ln p + 7.91\sqrt{p}.$$

(a) Use a graphing utility to plot the data and graph the model.

(b) Find the rate of change of T with respect to p when $p = 10$ and $p = 70$.

(c) Use a graphing utility to graph T'. Find $\lim_{p \to \infty} T'(p)$ and interpret the result in the context of the problem.

108. ***Modeling Data*** The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is one atmosphere (1.033227 kilograms per square centimeter). The table shows the pressures p (in atmospheres) at selected altitudes h (in kilometers).

h	0	5	10	15	20	25
p	1	0.55	0.25	0.12	0.06	0.02

(a) Use a graphing utility to find a model of the form $p = a + b \ln h$ for the data. Explain why the result is an error message.

(b) Use a graphing utility to find the logarithmic model $h = a + b \ln p$ for the data.

(c) Use a graphing utility to plot the data and graph the model.

(d) Use the model to estimate the altitude when $p = 0.75$.

(e) Use the model to estimate the pressure when $h = 13$.

(f) Use the model to find the rate of change of pressure when $h = 5$ and $h = 20$. Interpret the results.

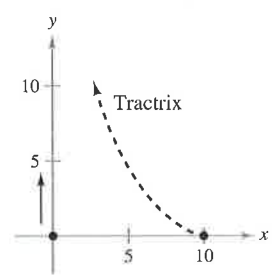
109. ***Tractrix*** A person walking along a dock drags a boat by a 10-meter rope. The boat travels along a path known as a *tractrix* (see figure). The equation of this path is

$$y = 10 \ln\left(\frac{10 + \sqrt{100 - x^2}}{x}\right) - \sqrt{100 - x^2}.$$

(a) Use a graphing utility to graph the function.

(b) What is the slope of this path when $x = 5$ and $x = 9$?

(c) What does the slope of the path approach as $x \to 10$?



110. ***Conjecture*** Use a graphing utility to graph f and g in the same viewing window and determine which is increasing at the greater rate for large values of x. What can you conclude about the rate of growth of the natural logarithmic function?

(a) $f(x) = \ln x, \quad g(x) = \sqrt{x}$ (b) $f(x) = \ln x, \quad g(x) = \sqrt[4]{x}$

111. Prove that the natural logarithmic function is one-to-one.

112. (a) Use a graphing utility to graph $y = \sqrt{x} - 4 \ln x$.

(b) Use the graph to identify any relative minima and inflection points.

(c) Use calculus to verify your answer to part (b).

Integrals to which the Log Rule can be applied often appear in disguised form. For instance, if a rational function has a *numerator of degree greater than or equal to that of the denominator*, division may reveal a form to which you can apply the Log Rule. This is shown in Example 5.

EXAMPLE 5 Using Long Division Before Integrating

Find $\displaystyle\int \frac{x^2+x+1}{x^2+1}\,dx.$

Solution Begin by using long division to rewrite the integrand.

$$\frac{x^2+x+1}{x^2+1} \quad\Longrightarrow\quad \begin{array}{r|l} & 1 \\ \hline x^2+1 & x^2+x+1 \\ & x^2+1 \\ \hline & x \end{array} \quad\Longrightarrow\quad 1+\frac{x}{x^2+1}$$

Now, you can integrate to obtain

$$\begin{aligned} \int \frac{x^2+x+1}{x^2+1}\,dx &= \int\left(1+\frac{x}{x^2+1}\right)dx && \text{Rewrite using long division.} \\ &= \int dx + \frac{1}{2}\int\frac{2x}{x^2+1}\,dx && \text{Rewrite as two integrals.} \\ &= x + \frac{1}{2}\ln(x^2+1) + C. && \text{Integrate.} \end{aligned}$$

Check this result by differentiating to obtain the original integrand.

The next example gives another instance in which the use of the Log Rule is disguised. In this case, a change of variables helps you recognize the Log Rule.

EXAMPLE 6 Change of Variables with the Log Rule

Find $\displaystyle\int \frac{2x}{(x+1)^2}\,dx.$

Solution If you let $u = x + 1$, then $du = dx$ and $x = u - 1$.

$$\begin{aligned} \int \frac{2x}{(x+1)^2}\,dx &= \int \frac{2(u-1)}{u^2}\,du && \text{Substitute.} \\ &= 2\int\left(\frac{u}{u^2}-\frac{1}{u^2}\right)du && \text{Rewrite as two fractions.} \\ &= 2\int\frac{du}{u} - 2\int u^{-2}\,du && \text{Rewrite as two integrals.} \\ &= 2\ln|u| - 2\left(\frac{u^{-1}}{-1}\right) + C && \text{Integrate.} \\ &= 2\ln|u| + \frac{2}{u} + C && \text{Simplify.} \\ &= 2\ln|x+1| + \frac{2}{x+1} + C && \text{Back-substitute.} \end{aligned}$$

Check this result by differentiating to obtain the original integrand.

TECHNOLOGY If you have access to a computer algebra system, use it to find the indefinite integrals in Examples 5 and 6. How does the form of the antiderivative that it gives you compare with that given in Examples 5 and 6?

As you study the methods shown in Examples 5 and 6, be aware that both methods involve rewriting a disguised integrand so that it fits one or more of the basic integration formulas. Throughout the remaining sections of Chapter 5 and in Chapter 8, much time will be devoted to integration techniques. To master these techniques, you must recognize the "form-fitting" nature of integration. In this sense, integration is not nearly as straightforward as differentiation. Differentiation takes the form

"Here is the question; what is the answer?"

Integration is more like

"Here is the answer; what is the question?"

The following are guidelines you can use for integration.

STUDY TIP Keep in mind that you can check your answer to an integration problem by differentiating the answer. For instance, in Example 7, the derivative of $y = \ln|\ln x| + C$ is $y' = 1/(x \ln x)$.

Guidelines for Integration

1. **Learn a basic list of integration formulas. (Including those given in this section, you now have 12 formulas: the Power Rule, the Log Rule, and ten trigonometric rules. By the end of Section 5.7, this list will have expanded to 20 basic rules.)**
2. **Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula.**
3. **If you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, or addition and subtraction of the same quantity. Be creative.**
4. **If you have access to computer software that will find antiderivatives symbolically, use it.**

EXAMPLE 7 u-Substitution and the Log Rule

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{1}{x \ln x}$.

Solution The solution can be written as an indefinite integral.

$$y = \int \frac{1}{x \ln x}\, dx$$

Because the integrand is a quotient whose denominator is raised to the first power, you should try the Log Rule. There are three basic choices for u. The choices $u = x$ and $u = x \ln x$ fail to fit the u'/u form of the Log Rule. However, the third choice does fit. Letting $u = \ln x$ produces $u' = 1/x$, and you obtain the following.

$$\int \frac{1}{x \ln x}\, dx = \int \frac{1/x}{\ln x}\, dx \qquad \text{Divide numerator and denominator by } x.$$

$$= \int \frac{u'}{u}\, dx \qquad \text{Substitute: } u = \ln x.$$

$$= \ln|u| + C \qquad \text{Apply Log Rule.}$$

$$= \ln|\ln x| + C \qquad \text{Back-substitute.}$$

So, the solution is $y = \ln|\ln x| + C$.

Integrals of Trigonometric Functions

In Section 4.1, you looked at six trigonometric integration rules—the six that correspond directly to differentiation rules. With the Log Rule, you can now complete the set of basic trigonometric integration formulas.

EXAMPLE 8 Using a Trigonometric Identity

Find $\int \tan x \, dx$.

Solution This integral does not seem to fit any formulas on our basic list. However, by using a trigonometric identity, you obtain

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx.$$

Knowing that $D_x[\cos x] = -\sin x$, you can let $u = \cos x$ and write

$$\begin{aligned}
\int \tan x \, dx &= -\int \frac{-\sin x}{\cos x} \, dx && \text{Trigonometric identity} \\
&= -\int \frac{u'}{u} \, dx && \text{Substitute: } u = \cos x. \\
&= -\ln|u| + C && \text{Apply Log Rule.} \\
&= -\ln|\cos x| + C. && \text{Back-substitute.}
\end{aligned}$$

Example 8 uses a trigonometric identity to derive an integration rule for the tangent function. The next example takes a rather unusual step (multiplying and dividing by the same quantity) to derive an integration rule for the secant function.

EXAMPLE 9 Derivation of the Secant Formula

Find $\int \sec x \, dx$.

Solution Consider the following procedure.

$$\begin{aligned}
\int \sec x \, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx
\end{aligned}$$

Letting u be the denominator of this quotient produces

$$u = \sec x + \tan x \quad \Rightarrow \quad u' = \sec x \tan x + \sec^2 x.$$

So, you can conclude that

$$\begin{aligned}
\int \sec x \, dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx && \text{Rewrite integrand.} \\
&= \int \frac{u'}{u} \, dx && \text{Substitute: } u = \sec x + \tan x. \\
&= \ln|u| + C && \text{Apply Log Rule.} \\
&= \ln|\sec x + \tan x| + C. && \text{Back-substitute.}
\end{aligned}$$

With the results of Examples 8 and 9, you now have integration formulas for $\sin x$, $\cos x$, $\tan x$, and $\sec x$. All six trigonometric rules are summarized below.

NOTE Using trigonometric identities and properties of logarithms, you could rewrite these six integration rules in other forms. For instance, you could write

$$\int \csc u\,du = \ln|\csc u - \cot u| + C.$$

(See Exercises 83–86.)

Integrals of the Six Basic Trigonometric Functions

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \tan u\,du = -\ln|\cos u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

EXAMPLE 10 Integrating Trigonometric Functions

Evaluate $\displaystyle\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx$.

Solution Using $1 + \tan^2 x = \sec^2 x$, you can write

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1 + \tan^2 x}\,dx &= \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx \\
&= \int_0^{\pi/4} \sec x\,dx && \sec x \geq 0 \text{ for } 0 \leq x \leq \frac{\pi}{4}. \\
&= \ln|\sec x + \tan x|\Big]_0^{\pi/4} \\
&= \ln\left(\sqrt{2} + 1\right) - \ln 1 \\
&\approx 0.881.
\end{aligned}$$

EXAMPLE 11 Finding an Average Value

Find the average value of $f(x) = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$.

Solution

$$\begin{aligned}
\text{Average value} &= \frac{1}{(\pi/4) - 0}\int_0^{\pi/4} \tan x\,dx && \text{Average value} = \frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{4}{\pi}\int_0^{\pi/4} \tan x\,dx && \text{Simplify.} \\
&= \frac{4}{\pi}\Big[-\ln|\cos x|\Big]_0^{\pi/4} && \text{Integrate.} \\
&= -\frac{4}{\pi}\left[\ln\left(\frac{\sqrt{2}}{2}\right) - \ln(1)\right] \\
&= -\frac{4}{\pi}\ln\left(\frac{\sqrt{2}}{2}\right) \\
&\approx 0.441
\end{aligned}$$

The average value is about 0.441, as shown in Figure 5.9.

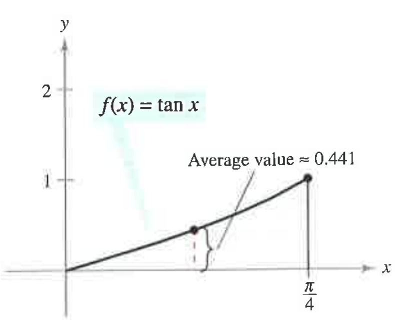


Figure 5.9

Exercises for Section 5.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–24, find the indefinite integral.

1. $\int \frac{5}{x}\,dx$
2. $\int \frac{10}{x}\,dx$
3. $\int \frac{1}{x+1}\,dx$
4. $\int \frac{1}{x-5}\,dx$
5. $\int \frac{1}{3-2x}\,dx$
6. $\int \frac{1}{3x+2}\,dx$
7. $\int \frac{x}{x^2+1}\,dx$
8. $\int \frac{x^2}{3-x^3}\,dx$
9. $\int \frac{x^2-4}{x}\,dx$
10. $\int \frac{x}{\sqrt{9-x^2}}\,dx$
11. $\int \frac{x^2+2x+3}{x^3+3x^2+9x}\,dx$
12. $\int \frac{x(x+2)}{x^3+3x^2-4}\,dx$
13. $\int \frac{x^2-3x+2}{x+1}\,dx$
14. $\int \frac{2x^2+7x-3}{x-2}\,dx$
15. $\int \frac{x^3-3x^2+5}{x-3}\,dx$
16. $\int \frac{x^3-6x-20}{x+5}\,dx$
17. $\int \frac{x^4+x-4}{x^2+2}\,dx$
18. $\int \frac{x^3-3x^2+4x-9}{x^2+3}\,dx$
19. $\int \frac{(\ln x)^2}{x}\,dx$
20. $\int \frac{1}{x\ln(x^3)}\,dx$
21. $\int \frac{1}{\sqrt{x}+1}\,dx$
22. $\int \frac{1}{x^{2/3}(1+x^{1/3})}\,dx$
23. $\int \frac{2x}{(x-1)^2}\,dx$
24. $\int \frac{x(x-2)}{(x-1)^3}\,dx$

In Exercises 25–28, find the indefinite integral by *u*-substitution. (*Hint:* Let *u* be the denominator of the integrand.)

25. $\int \frac{1}{1+\sqrt{2x}}\,dx$
26. $\int \frac{1}{1+\sqrt{3x}}\,dx$
27. $\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx$
28. $\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx$

In Exercises 29–36, find the indefinite integral.

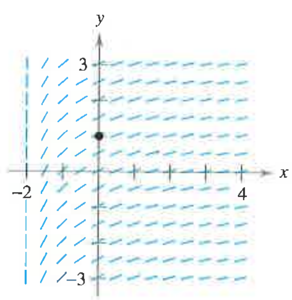
29. $\int \frac{\cos\theta}{\sin\theta}\,d\theta$
30. $\int \tan 5\theta\,d\theta$
31. $\int \csc 2x\,dx$
32. $\int \sec\frac{x}{2}\,dx$
33. $\int \frac{\cos t}{1+\sin t}\,dt$
34. $\int \frac{\csc^2 t}{\cot t}\,dt$
35. $\int \frac{\sec x\tan x}{\sec x - 1}\,dx$
36. $\int (\sec t + \tan t)\,dt$

In Exercises 37–40, solve the differential equation. Use a graphing utility to graph three solutions, one of which passes through the given point.

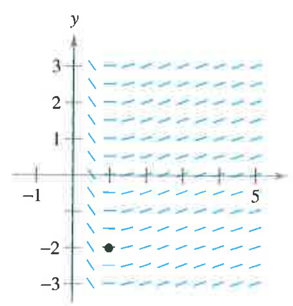
37. $\frac{dy}{dx} = \frac{3}{2-x}$, $(1, 0)$
38. $\frac{dy}{dx} = \frac{2x}{x^2-9}$, $(0, 4)$
39. $\frac{ds}{d\theta} = \tan 2\theta$, $(0, 2)$
40. $\frac{dr}{dt} = \frac{\sec^2 t}{\tan t + 1}$, $(\pi, 4)$
41. Determine the function f if $f''(x) = \frac{2}{x^2}$, $f(1) = 1$, $f'(1) = 1$, $x > 0$.
42. Determine the function f if $f''(x) = -\frac{4}{(x-1)^2} - 2$, $f(2) = 3$, $f'(2) = 0$, $x > 1$.

Slope Fields **In Exercises 43–46, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

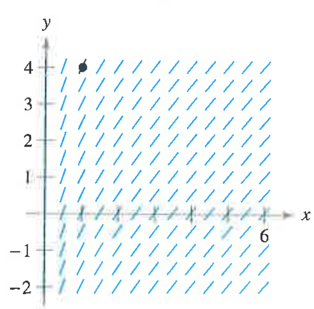
43. $\frac{dy}{dx} = \frac{1}{x+2}$, $(0, 1)$



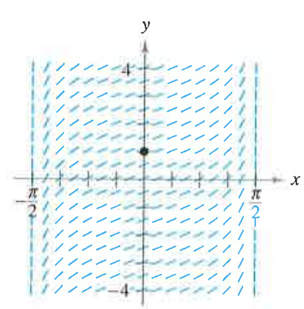
44. $\frac{dy}{dx} = \frac{\ln x}{x}$, $(1, -2)$



45. $\frac{dy}{dx} = 1 + \frac{1}{x}$, $(1, 4)$



46. $\frac{dy}{dx} = \sec x$, $(0, 1)$



In Exercises 47–54, evaluate the definite integral. Use a graphing utility to verify your result.

47. $\displaystyle\int_0^4 \frac{5}{3x+1}\,dx$

48. $\displaystyle\int_{-1}^1 \frac{1}{x+2}\,dx$

49. $\displaystyle\int_1^e \frac{(1+\ln x)^2}{x}\,dx$

50. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx$

51. $\displaystyle\int_0^2 \frac{x^2-2}{x+1}\,dx$

52. $\displaystyle\int_0^1 \frac{x-1}{x+1}\,dx$

53. $\displaystyle\int_1^2 \frac{1-\cos\theta}{\theta-\sin\theta}\,d\theta$

54. $\displaystyle\int_{0.1}^{0.2} (\csc 2\theta - \cot 2\theta)^2\,d\theta$

In Exercises 55–60, use a computer algebra system to find or evaluate the integral.

55. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$

56. $\displaystyle\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

57. $\displaystyle\int \frac{\sqrt{x}}{x-1}\,dx$

58. $\displaystyle\int \frac{x^2}{x-1}\,dx$

59. $\displaystyle\int_{\pi/4}^{\pi/2} (\csc x - \sin x)\,dx$

60. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{\sin^2 x - \cos^2 x}{\cos x}\,dx$

In Exercises 61–64, find $F'(x)$.

61. $\displaystyle F(x) = \int_1^x \frac{1}{t}\,dt$

62. $\displaystyle F(x) = \int_0^x \tan t\,dt$

63. $\displaystyle F(x) = \int_1^{3x} \frac{1}{t}\,dt$

64. $\displaystyle F(x) = \int_1^{x^2} \frac{1}{t}\,dt$

Approximation **In Exercises 65 and 66, determine which value best approximates the area of the region between the x-axis and the graph of the function over the given interval. (Make your selection on the basis of a sketch of the region and not by performing any calculations.)**

65. $f(x) = \sec x$, $[0, 1]$

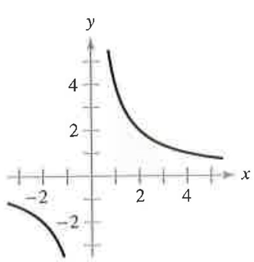
(a) 6 (b) -6 (c) $\frac{1}{2}$ (d) 1.25 (e) 3

66. $f(x) = \dfrac{2x}{x^2+1}$, $[0, 4]$

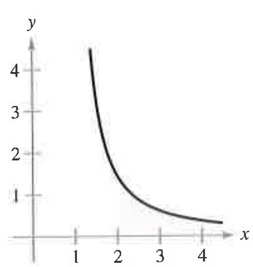
(a) 3 (b) 7 (c) -2 (d) 5 (e) 1

Area **In Exercises 67–70, find the area of the given region. Use a graphing utility to verify your result.**

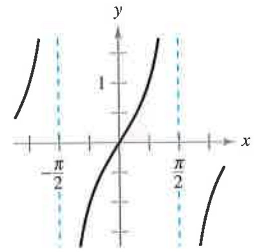
67. $y = \dfrac{4}{x}$



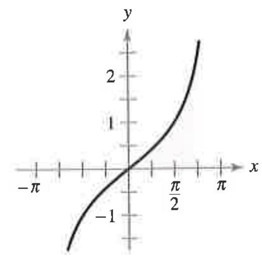
68. $y = \dfrac{2}{x\ln x}$



69. $y = \tan x$



70. $y = \dfrac{\sin x}{1+\cos x}$



Area **In Exercises 71–74, find the area of the region bounded by the graphs of the equations. Use a graphing utility to verify your result.**

71. $y = \dfrac{x^2+4}{x}$, $x = 1$, $x = 4$, $y = 0$

72. $y = \dfrac{x+4}{x}$, $x = 1$, $x = 4$, $y = 0$

73. $y = 2\sec\dfrac{\pi x}{6}$, $x = 0$, $x = 2$, $y = 0$

74. $y = 2x - \tan(0.3x)$, $x = 1$, $x = 4$, $y = 0$

Numerical Integration **In Exercises 75–78, use the Trapezoidal Rule and Simpson's Rule to approximate the value of the definite integral. Let $n = 4$ and round your answer to four decimal places. Use a graphing utility to verify your result.**

75. $\displaystyle\int_1^5 \frac{12}{x}\,dx$

76. $\displaystyle\int_0^4 \frac{8x}{x^2+4}\,dx$

77. $\displaystyle\int_2^6 \ln x\,dx$

78. $\displaystyle\int_{-\pi/3}^{\pi/3} \sec x\,dx$

Writing About Concepts

In Exercises 79–82, state the integration formula you would use to perform the integration. Do not integrate.

79. $\displaystyle\int \sqrt[3]{x}\,dx$

80. $\displaystyle\int \frac{x}{(x^2+4)^3}\,dx$

81. $\displaystyle\int \frac{x}{x^2+4}\,dx$

82. $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx$

In Exercises 83–86, show that the two formulas are equivalent.

83. $\int \tan x\, dx = -\ln|\cos x| + C$

$\int \tan x\, dx = \ln|\sec x| + C$

84. $\int \cot x\, dx = \ln|\sin x| + C$

$\int \cot x\, dx = -\ln|\csc x| + C$

85. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$

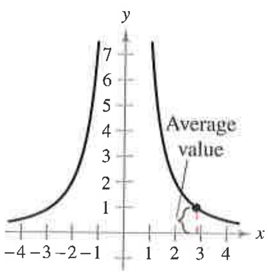
$\int \sec x\, dx = -\ln|\sec x - \tan x| + C$

86. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$

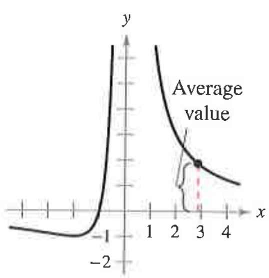
$\int \csc x\, dx = \ln|\csc x - \cot x| + C$

In Exercises 87–90, find the average value of the function over the given interval.

87. $f(x) = \dfrac{8}{x^2}$, $[2, 4]$



88. $f(x) = \dfrac{4(x + 1)}{x^2}$, $[2, 4]$



89. $f(x) = \dfrac{\ln x}{x}$, $[1, e]$

90. $f(x) = \sec \dfrac{\pi x}{6}$, $[0, 2]$

91. ***Population Growth*** A population of bacteria is changing at a rate of

$$\frac{dP}{dt} = \frac{3000}{1 + 0.25t}$$

where t is the time in days. The initial population (when $t = 0$) is 1000. Write an equation that gives the population at any time t, and find the population when $t = 3$ days.

92. ***Heat Transfer*** Find the time required for an object to cool from 300°F to 250°F by evaluating

$$t = \frac{10}{\ln 2}\int_{250}^{300} \frac{1}{T - 100}\, dT$$

where t is time in minutes.

93. ***Average Price*** The demand equation for a product is

$$p = \frac{90{,}000}{400 + 3x}.$$

Find the *average* price p on the interval $40 \le x \le 50$.

94. ***Sales*** The rate of change in sales S is inversely proportional to time t $(t > 1)$ measured in weeks. Find S as a function of t if sales after 2 and 4 weeks are 200 units and 300 units, respectively.

95. ***Orthogonal Trajectory***

(a) Use a graphing utility to graph the equation $2x^2 - y^2 = 8$.

(b) Evaluate the integral to find y^2 in terms of x.

$$y^2 = e^{-\int (1/x)\, dx}$$

For a particular value of the constant of integration, graph the result in the same viewing window used in part (a).

(c) Verify that the tangents to the graphs of parts (a) and (b) are perpendicular at the points of intersection.

96. Graph the function

$$f_k(x) = \frac{x^k - 1}{k}$$

for $k = 1, 0.5$, and 0.1 on $[0, 10]$. Find $\lim_{k\to 0^+} f_k(x)$.

True or False? **In Exercises 97–100, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

97. $(\ln x)^{1/2} = \frac{1}{2}(\ln x)$

98. $\int \ln x\, dx = (1/x) + C$

99. $\int \frac{1}{x}\, dx = \ln|cx|, \quad c \neq 0$

100. $\int_{-1}^{2} \frac{1}{x}\, dx = \Big[\ln|x|\Big]_{-1}^{2} = \ln 2 - \ln 1 = \ln 2$

101. Graph the function

$$f(x) = \frac{x}{1 + x^2}$$

on the interval $[0, \infty)$.

(a) Find the area bounded by the graph of f and the line $y = \frac{1}{2}x$.

(b) Determine the values of the slope m such that the line $y = mx$ and the graph of f enclose a finite region.

(c) Calculate the area of this region as a function of m.

102. Prove that the function

$$F(x) = \int_{x}^{2x} \frac{1}{t}\, dt$$

is constant on the interval $(0, \infty)$.

Section 5.3

Inverse Functions

- **Verify that one function is the inverse function of another function.**
- **Determine whether a function has an inverse function.**
- **Find the derivative of an inverse function.**

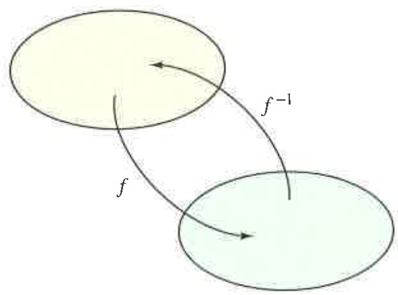
Inverse Functions

Recall from Section P.3 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 3$ from $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ can be written as

$$f: \{(1, 4), (2, 5), (3, 6), (4, 7)\}.$$

By interchanging the first and second coordinates of each ordered pair, you can form the **inverse function** of f. This function is denoted by f^{-1}. It is a function from B to A, and can be written as

$$f^{-1}: \{(4, 1), (5, 2), (6, 3), (7, 4)\}.$$



Domain of f = range of f^{-1}
Domain of f^{-1} = range of f
Figure 5.10

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 5.10. The functions f and f^{-1} have the effect of "undoing" each other. That is, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x$$

EXPLORATION

Finding Inverse Functions Explain how to "undo" each of the following functions. Then use your explanation to write the inverse function of f.

a. $f(x) = x - 5$

b. $f(x) = 6x$

c. $f(x) = \dfrac{x}{2}$

d. $f(x) = 3x + 2$

e. $f(x) = x^3$

f. $f(x) = 4(x - 2)$

Use a graphing utility to graph each function and its inverse function in the same "square" viewing window. What observation can you make about each pair of graphs?

Definition of Inverse Function

A function g is the **inverse function** of the function f if

$$f(g(x)) = x \quad \text{for each } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \quad \text{for each } x \text{ in the domain of } f.$$

The function g is denoted by f^{-1} (read "f inverse").

NOTE Although the notation used to denote an inverse function resembles *exponential notation*, it is a different use of -1 as a superscript. That is, in general, $f^{-1}(x) \neq 1/f(x)$.

Here are some important observations about inverse functions.

1. If g is the inverse function of f, then f is the inverse function of g.
2. The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
3. A function need not have an inverse function, but if it does, the inverse function is unique (see Exercise 99).

You can think of f^{-1} as undoing what has been done by f. For example, subtraction can be used to undo addition, and division can be used to undo multiplication. Use the definition of an inverse function to check the following.

$$f(x) = x + c \quad \text{and} \quad f^{-1}(x) = x - c \quad \text{are inverse functions of each other.}$$

$$f(x) = cx \quad \text{and} \quad f^{-1}(x) = \frac{x}{c},\ c \neq 0, \quad \text{are inverse functions of each other.}$$

EXAMPLE 1 Verifying Inverse Functions

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

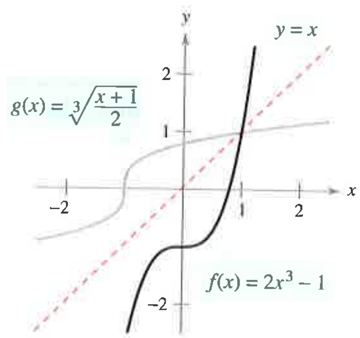
Solution Because the domains and ranges of both f and g consist of all real numbers, you can conclude that both composite functions exist for all x. The composition of f with g is given by

$$\begin{aligned} f(g(x)) &= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x + 1 - 1 \\ &= x. \end{aligned}$$

The composition of g with f is given by

$$\begin{aligned} g(f(x)) &= \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} \\ &= \sqrt[3]{\frac{2x^3}{2}} \\ &= \sqrt[3]{x^3} \\ &= x. \end{aligned}$$

Because $f(g(x)) = x$ and $g(f(x)) = x$, you can conclude that f and g are inverse functions of each other (see Figure 5.11).



f and g are inverse functions of each other.
Figure 5.11

STUDY TIP In Example 1, try comparing the functions f and g verbally.

For f: First cube x, then multiply by 2, then subtract 1.

For g: First add 1, then divide by 2, then take the cube root.

Do you see the "undoing pattern"?

In Figure 5.11, the graphs of f and $g = f^{-1}$ appear to be mirror images of each other with respect to the line $y = x$. The graph of f^{-1} is a **reflection** of the graph of f in the line $y = x$. This idea is generalized in the following theorem.

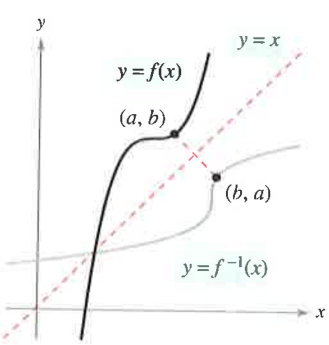
THEOREM 5.6 Reflective Property of Inverse Functions

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

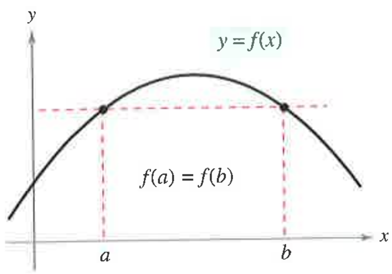
Proof If (a, b) is on the graph of f, then $f(a) = b$ and you can write

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

So, (b, a) is on the graph of f^{-1}, as shown in Figure 5.12. A similar argument will prove the theorem in the other direction.



The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.
Figure 5.12



If a horizontal line intersects the graph of f twice, then f is not one-to-one.
Figure 5.13

Existence of an Inverse Function

Not every function has an inverse function, and Theorem 5.6 suggests a graphical test for those that do—the **Horizontal Line Test** for an inverse function. This test states that a function f has an inverse function if and only if every horizontal line intersects the graph of f at most once (see Figure 5.13). The following theorem formally states why the horizontal line test is valid. (Recall from Section 3.3 that a function is *strictly monotonic* if it is either increasing on its entire domain or decreasing on its entire domain.)

THEOREM 5.7 The Existence of an Inverse Function

1. A function has an inverse function if and only if it is one-to-one.
2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

Proof To prove the second part of the theorem, recall from Section P.3 that f is one-to-one if for x_1 and x_2 in its domain

$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2.$$

The *contrapositive* of this implication is logically equivalent and states that

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

Now, choose x_1 and x_2 in the domain of f. If $x_1 \neq x_2$, then, because f is strictly monotonic, it follows that either

$$f(x_1) < f(x_2) \qquad \text{or} \qquad f(x_1) > f(x_2).$$

In either case, $f(x_1) \neq f(x_2)$. So, f is one-to-one on the interval. The proof of the first part of the theorem is left as an exercise (see Exercise 100).

EXAMPLE 2 The Existence of an Inverse Function

Which of the functions has an inverse function?

a. $f(x) = x^3 + x - 1$ **b.** $f(x) = x^3 - x + 1$

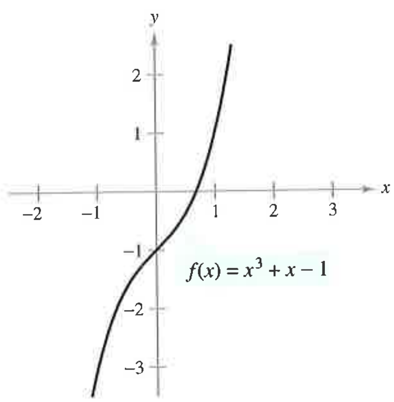
Solution

a. From the graph of f shown in Figure 5.14(a), it appears that f is increasing over its entire domain. To verify this, note that the derivative, $f'(x) = 3x^2 + 1$, is positive for all real values of x. So, f is strictly monotonic and it must have an inverse function.

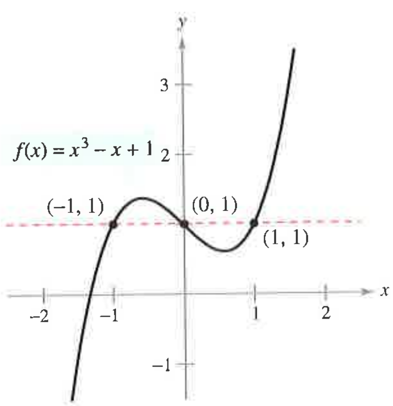
b. From the graph of f shown in Figure 5.14(b), you can see that the function does not pass the horizontal line test. In other words, it is not one-to-one. For instance, f has the same value when $x = -1$, 0, and 1.

$$f(-1) = f(1) = f(0) = 1 \qquad \text{Not one-to-one}$$

So, by Theorem 5.7, f does not have an inverse function.



(a) Because f is increasing over its entire domain, it has an inverse function.



(b) Because f is not one-to-one, it does not have an inverse function.
Figure 5.14

NOTE Often it is easier to prove that a function *has* an inverse function than to find the inverse function. For instance, it would be difficult algebraically to find the inverse function of the function in Example 2(a).

The following guidelines suggest a procedure for finding an inverse function.

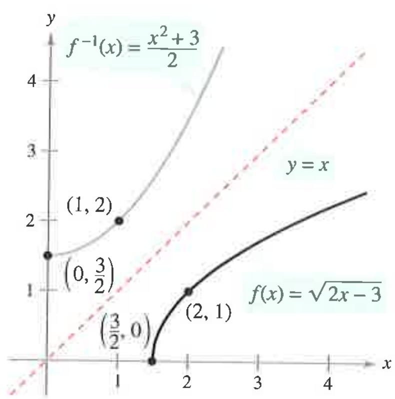
Guidelines for Finding an Inverse Function

1. Use Theorem 5.7 to determine whether the function given by $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} to be the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

EXAMPLE 3 Finding an Inverse Function

Find the inverse function of

$$f(x) = \sqrt{2x - 3}.$$



The domain of f^{-1}, $[0, \infty)$, is the range of f.
Figure 5.15

Solution The function has an inverse function because it is increasing on its entire domain (see Figure 5.15). To find an equation for the inverse function, let $y = f(x)$ and solve for x in terms of y.

$$\sqrt{2x - 3} = y \qquad \text{Let } y = f(x).$$

$$2x - 3 = y^2 \qquad \text{Square each side.}$$

$$x = \frac{y^2 + 3}{2} \qquad \text{Solve for } x.$$

$$y = \frac{x^2 + 3}{2} \qquad \text{Interchange } x \text{ and } y.$$

$$f^{-1}(x) = \frac{x^2 + 3}{2} \qquad \text{Replace } y \text{ by } f^{-1}(x).$$

The domain of f^{-1} is the range of f, which is $[0, \infty)$. You can verify this result as shown.

$$f(f^{-1}(x)) = \sqrt{2\left(\frac{x^2 + 3}{2}\right) - 3} = \sqrt{x^2} = x, \quad x \geq 0$$

$$f^{-1}(f(x)) = \frac{\left(\sqrt{2x - 3}\right)^2 + 3}{2} = \frac{2x - 3 + 3}{2} = x, \quad x \geq \frac{3}{2}$$

NOTE Remember that any letter can be used to represent the independent variable. So,

$$f^{-1}(y) = \frac{y^2 + 3}{2}$$

$$f^{-1}(x) = \frac{x^2 + 3}{2}$$

$$f^{-1}(s) = \frac{s^2 + 3}{2}$$

all represent the same function.

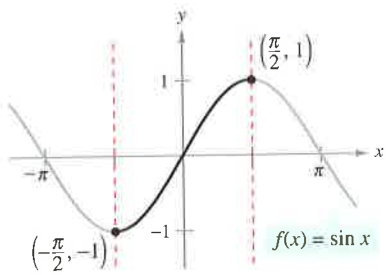
Theorem 5.7 is useful in the following type of problem. Suppose you are given a function that is *not* one-to-one on its domain. By restricting the domain to an interval on which the function is strictly monotonic, you can conclude that the new function *is* one-to-one on the restricted domain.

EXAMPLE 4 Testing Whether a Function Is One-to-One

Show that the sine function

$$f(x) = \sin x$$

is not one-to-one on the entire real line. Then show that $[-\pi/2, \pi/2]$ is the largest interval, centered at the origin, for which f is strictly monotonic.



f is one-to-one on the interval $[-\pi/2, \pi/2]$.
Figure 5.16

Solution It is clear that f is not one-to-one, because many different x-values yield the same y-value. For instance,

$$\sin(0) = 0 = \sin(\pi).$$

Moreover, f is increasing on the open interval $(-\pi/2, \pi/2)$, because its derivative

$$f'(x) = \cos x$$

is positive there. Finally, because the left and right endpoints correspond to relative extrema of the sine function, you can conclude that f is increasing on the closed interval $[-\pi/2, \pi/2]$ *and* that in any larger interval the function is not strictly monotonic (see Figure 5.16).

Derivative of an Inverse Function

The next two theorems discuss the derivative of an inverse function. The reasonableness of Theorem 5.8 follows from the reflective property of inverse functions as shown in Figure 5.12. Proofs of the two theorems are given in Appendix A.

THEOREM 5.8 Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

EXPLORATION

Graph the inverse functions

$$f(x) = x^3$$

and

$$g(x) = x^{1/3}.$$

Calculate the slope of f at $(1, 1)$, $(2, 8)$, and $(3, 27)$, and the slope of g at $(1, 1)$, $(8, 2)$, and $(27, 3)$. What do you observe? What happens at $(0, 0)$?

THEOREM 5.9 The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

EXAMPLE 5 Evaluating the Derivative of an Inverse Function

Let $f(x) = \frac{1}{4}x^3 + x - 1$.

a. What is the value of $f^{-1}(x)$ when $x = 3$?

b. What is the value of $(f^{-1})'(x)$ when $x = 3$?

Solution Notice that f is one-to-one and therefore has an inverse function.

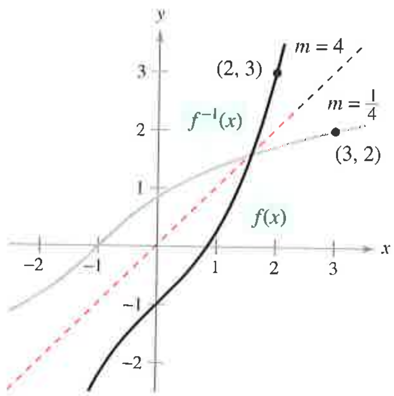
a. Because $f(x) = 3$ when $x = 2$, you know that $f^{-1}(3) = 2$.

b. Because the function f is differentiable and has an inverse function, you can apply Theorem 5.9 (with $g = f^{-1}$) to write

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)}.$$

Moreover, using $f'(x) = \frac{3}{4}x^2 + 1$, you can conclude that

$$(f^{-1})'(3) = \frac{1}{f'(2)} = \frac{1}{\frac{3}{4}(2^2) + 1} = \frac{1}{4}.$$



The graphs of the inverse functions f and f^{-1} have reciprocal slopes at points (a, b) and (b, a).
Figure 5.17

In Example 5, note that at the point (2, 3) the slope of the graph of f is 4 and at the point (3, 2) the slope of the graph of f^{-1} is $\frac{1}{4}$ (see Figure 5.17). This reciprocal relationship (which follows from Theorem 5.9) can be written as shown below. If $y = g(x) = f^{-1}(x)$, then $f(y) = x$ and $f'(y) = \frac{dx}{dy}$. Theorem 5.9 says that

$$g'(x) = \frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{f'(y)} = \frac{1}{(dx/dy)}.$$

So, $\displaystyle \frac{dy}{dx} = \frac{1}{dx/dy}$.

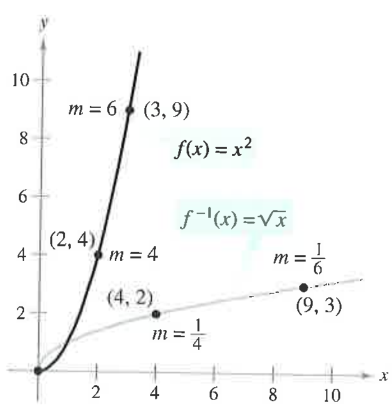
EXAMPLE 6 Graphs of Inverse Functions Have Reciprocal Slopes

Let $f(x) = x^2$ (for $x \geq 0$) and let $f^{-1}(x) = \sqrt{x}$. Show that the slopes of the graphs of f and f^{-1} are reciprocals at each of the following points.

a. (2, 4) and (4, 2) **b.** (3, 9) and (9, 3)

Solution The derivatives of f and f^{-1} are given by

$$f'(x) = 2x \quad \text{and} \quad (f^{-1})'(x) = \frac{1}{2\sqrt{x}}.$$



At (0, 0), the derivative of f is 0, and the derivative of f^{-1} does not exist.
Figure 5.18

a. At (2, 4), the slope of the graph of f is $f'(2) = 2(2) = 4$. At (4, 2), the slope of the graph of f^{-1} is

$$(f^{-1})'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}.$$

b. At (3, 9), the slope of the graph of f is $f'(3) = 2(3) = 6$. At (9, 3), the slope of the graph of f^{-1} is

$$(f^{-1})'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{2(3)} = \frac{1}{6}.$$

So, in both cases, the slopes are reciprocals, as shown in Figure 5.18.

Exercises for Section 5.3

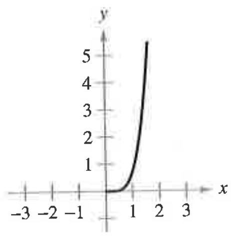
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, show that f and g are inverse functions (a) analytically and (b) graphically.

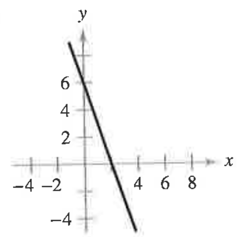
1. $f(x) = 5x + 1,\quad g(x) = (x - 1)/5$
2. $f(x) = 3 - 4x,\quad g(x) = (3 - x)/4$
3. $f(x) = x^3,\quad g(x) = \sqrt[3]{x}$
4. $f(x) = 1 - x^3,\quad g(x) = \sqrt[3]{1 - x}$
5. $f(x) = \sqrt{x - 4},\quad g(x) = x^2 + 4,\ \ x \ge 0$
6. $f(x) = 16 - x^2,\ \ x \ge 0,\quad g(x) = \sqrt{16 - x}$
7. $f(x) = 1/x,\quad g(x) = 1/x$
8. $f(x) = \dfrac{1}{1 + x},\ \ x \ge 0,\quad g(x) = \dfrac{1 - x}{x},\ \ 0 < x \le 1$

In Exercises 9–12, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]

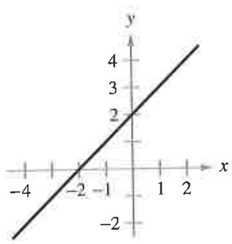
(a)



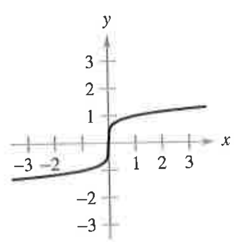
(b)



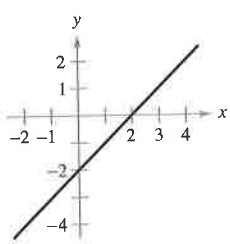
(c)



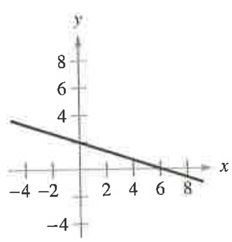
(d)



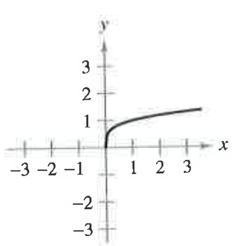
9.



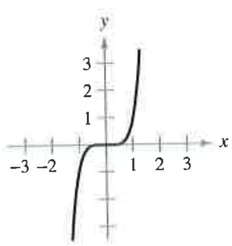
10.



11.

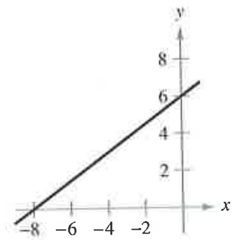


12.

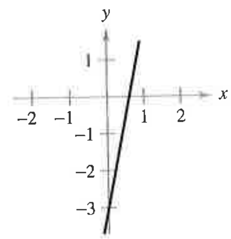


In Exercises 13–16, use the Horizontal Line Test to determine whether the function is one-to-one on its entire domain and therefore has an inverse function. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.

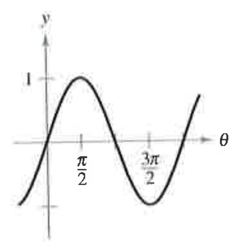
13. $f(x) = \frac{3}{4}x + 6$



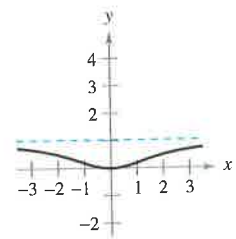
14. $f(x) = 5x - 3$



15. $f(\theta) = \sin\theta$



16. $f(x) = \dfrac{x^2}{x^2 + 4}$



In Exercises 17–22, use a graphing utility to graph the function. Determine whether the function is one-to-one on its entire domain.

17. $h(s) = \dfrac{1}{s - 2} - 3$
18. $g(t) = \dfrac{1}{\sqrt{t^2 + 1}}$
19. $f(x) = \ln x$
20. $f(x) = 5x\sqrt{x - 1}$
21. $g(x) = (x + 5)^3$
22. $h(x) = |x + 4| - |x - 4|$

In Exercises 23–28, use the derivative to determine whether the function is strictly monotonic on its entire domain and therefore has an inverse function.

23. $f(x) = \ln(x - 3)$
24. $f(x) = \cos\dfrac{3x}{2}$
25. $f(x) = \dfrac{x^4}{4} - 2x^2$
26. $f(x) = x^3 - 6x^2 + 12x$
27. $f(x) = 2 - x - x^3$
28. $f(x) = (x + a)^3 + b$

In Exercises 29–36, find the inverse function of f. Graph (by hand) f and f^{-1}. Describe the relationship between the graphs.

29. $f(x) = 2x - 3$
30. $f(x) = 3x$
31. $f(x) = x^5$
32. $f(x) = x^3 - 1$
33. $f(x) = \sqrt{x}$
34. $f(x) = x^2,\ \ x \ge 0$
35. $f(x) = \sqrt{4 - x^2},\ \ x \ge 0$
36. $f(x) = \sqrt{x^2 - 4},\ \ x \ge 2$

Section 5.4 Exponential Functions: Differentiation and Integration

- **Develop properties of the natural exponential function.**
- **Differentiate natural exponential functions.**
- **Integrate natural exponential functions.**

The Natural Exponential Function

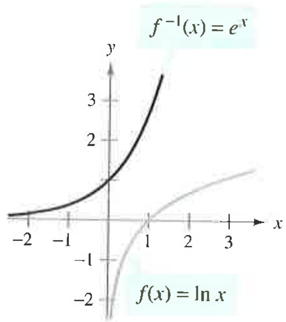
The function $f(x) = \ln x$ is increasing on its entire domain, and therefore it has an inverse function f^{-1}. The domain of f^{-1} is the set of all reals, and the range is the set of positive reals, as shown in Figure 5.19. So, for any real number x,

$$f(f^{-1}(x)) = \ln[f^{-1}(x)] = x. \qquad x \text{ is any real number.}$$

If x happens to be rational, then

$$\ln(e^x) = x \ln e = x(1) = x. \qquad x \text{ is a rational number.}$$

Because the natural logarithmic function is one-to-one, you can conclude that $f^{-1}(x)$ and e^x agree for *rational* values of x. The following definition extends the meaning of e^x to include *all* real values of x.



The inverse function of the natural logarithmic function is the natural exponential function.
Figure 5.19

Definition of the Natural Exponential Function

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

$$f^{-1}(x) = e^x.$$

That is,

$$y = e^x \quad \text{if and only if} \quad x = \ln y.$$

The Number e

The symbol e was first used by mathematician Leonhard Euler to represent the base of natural logarithms in a letter to another mathematician, Christian Goldbach, in 1731.

The inverse relationship between the natural logarithmic function and the natural exponential function can be summarized as follows.

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x \qquad \text{Inverse relationship}$$

EXAMPLE 1 Solving Exponential Equations

Solve $7 = e^{x+1}$.

Solution You can convert from exponential form to logarithmic form by *taking the natural logarithm of each side* of the equation.

$$
\begin{aligned}
7 &= e^{x+1} && \text{Write original equation.} \\
\ln 7 &= \ln(e^{x+1}) && \text{Take natural logarithm of each side.} \\
\ln 7 &= x + 1 && \text{Apply inverse property.} \\
-1 + \ln 7 &= x && \text{Solve for } x. \\
0.946 &\approx x && \text{Use a calculator.}
\end{aligned}
$$

Check this solution in the original equation.

EXAMPLE 2 Solving a Logarithmic Equation

Solve $\ln(2x - 3) = 5$.

Solution To convert from logarithmic form to exponential form, you can *exponentiate each side* of the logarithmic equation.

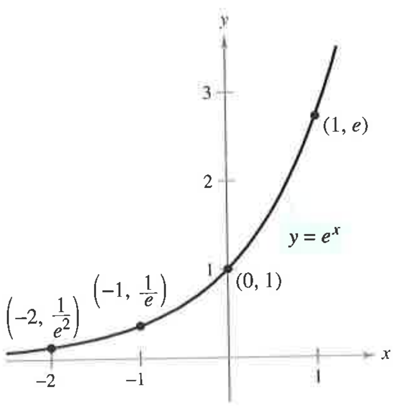
$$\ln(2x - 3) = 5 \qquad \text{Write original equation.}$$
$$e^{\ln(2x-3)} = e^5 \qquad \text{Exponentiate each side.}$$
$$2x - 3 = e^5 \qquad \text{Apply inverse property.}$$
$$x = \tfrac{1}{2}(e^5 + 3) \qquad \text{Solve for } x.$$
$$x \approx 75.707 \qquad \text{Use a calculator.}$$

The familiar rules for operating with rational exponents can be extended to the natural exponential function, as shown in the following theorem.

THEOREM 5.10 Operations with Exponential Functions

Let a and b be any real numbers.

1. $e^a e^b = e^{a+b}$
2. $\dfrac{e^a}{e^b} = e^{a-b}$



The natural exponential function is increasing, and its graph is concave upward.
Figure 5.20

Proof To prove Property 1, you can write

$$\ln(e^a e^b) = \ln(e^a) + \ln(e^b)$$
$$= a + b$$
$$= \ln(e^{a+b}).$$

Because the natural logarithmic function is one-to-one, you can conclude that

$$e^a e^b = e^{a+b}.$$

The proof of the second property is left to you (see Exercise 129).

In Section 5.3, you learned that an inverse function f^{-1} shares many properties with f. So, the natural exponential function inherits the following properties from the natural logarithmic function (see Figure 5.20).

Properties of the Natural Exponential Function

1. The domain of $f(x) = e^x$ is $(-\infty, \infty)$, and the range is $(0, \infty)$.
2. The function $f(x) = e^x$ is continuous, increasing, and one-to-one on its entire domain.
3. The graph of $f(x) = e^x$ is concave upward on its entire domain.
4. $\lim_{x \to -\infty} e^x = 0$ and $\lim_{x \to \infty} e^x = \infty$

Derivatives of Exponential Functions

One of the most intriguing (and useful) characteristics of the natural exponential function is that *it is its own derivative*. In other words, it is a solution to the differential equation $y' = y$. This result is stated in the next theorem.

FOR FURTHER INFORMATION To find out about derivatives of exponential functions of order 1/2, see the article "A Child's Garden of Fractional Derivatives" by Marcia Kleinz and Thomas J. Osler in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

THEOREM 5.11 Derivative of the Natural Exponential Function

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}[e^x] = e^x$
2. $\dfrac{d}{dx}[e^u] = e^u\dfrac{du}{dx}$

Proof To prove Property 1, use the fact that $\ln e^x = x$, and differentiate each side of the equation.

$$\ln e^x = x \qquad \text{Definition of exponential function}$$

$$\frac{d}{dx}[\ln e^x] = \frac{d}{dx}[x] \qquad \text{Differentiate each side with respect to } x.$$

$$\frac{1}{e^x}\frac{d}{dx}[e^x] = 1$$

$$\frac{d}{dx}[e^x] = e^x$$

The derivative of e^u follows from the Chain Rule.

NOTE You can interpret this theorem geometrically by saying that the slope of the graph of $f(x) = e^x$ at any point (x, e^x) is equal to the y-coordinate of the point.

EXAMPLE 3 Differentiating Exponential Functions

a. $\dfrac{d}{dx}[e^{2x-1}] = e^u\dfrac{du}{dx} = 2e^{2x-1}$ $\qquad u = 2x - 1$

b. $\dfrac{d}{dx}[e^{-3/x}] = e^u\dfrac{du}{dx} = \left(\dfrac{3}{x^2}\right)e^{-3/x} = \dfrac{3e^{-3/x}}{x^2}$ $\qquad u = -\dfrac{3}{x}$

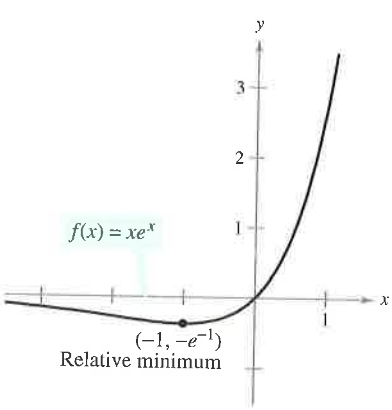
EXAMPLE 4 Locating Relative Extrema

Find the relative extrema of $f(x) = xe^x$.

Solution The derivative of f is given by

$$\begin{aligned} f'(x) &= x(e^x) + e^x(1) \qquad \text{Product Rule} \\ &= e^x(x+1). \end{aligned}$$

Because e^x is never 0, the derivative is 0 only when $x = -1$. Moreover, by the First Derivative Test, you can determine that this corresponds to a relative minimum, as shown in Figure 5.21. Because the derivative $f'(x) = e^x(x+1)$ is defined for all x, there are no other critical points.



The derivative of f changes from negative to positive at $x = -1$.
Figure 5.21

EXAMPLE 5 The Standard Normal Probability Density Function

Show that the *standard normal probability density function*

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

has points of inflection when $x = \pm 1$.

Solution To locate possible points of inflection, find the x-values for which the second derivative is 0.

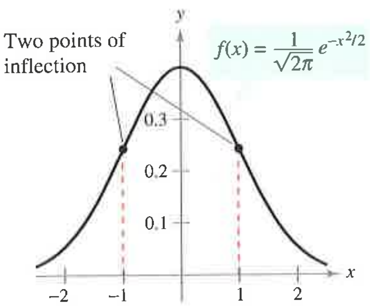
$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \qquad \text{Write original function.}$$

$$f'(x) = \frac{1}{\sqrt{2\pi}}(-x)e^{-x^2/2} \qquad \text{First derivative}$$

$$f''(x) = \frac{1}{\sqrt{2\pi}}[(-x)(-x)e^{-x^2/2} + (-1)e^{-x^2/2}] \qquad \text{Product Rule}$$

$$= \frac{1}{\sqrt{2\pi}}(e^{-x^2/2})(x^2 - 1) \qquad \text{Second derivative}$$

So, $f''(x) = 0$ when $x = \pm 1$, and you can apply the techniques of Chapter 3 to conclude that these values yield the two points of inflection shown in Figure 5.22.



The bell-shaped curve given by a standard normal probability density function
Figure 5.22

NOTE The general form of a normal probability density function (whose mean is 0) is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}$$

where σ is the standard deviation (σ is the lowercase Greek letter sigma). This "bell-shaped curve" has points of inflection when $x = \pm\sigma$.

EXAMPLE 6 Shares Traded

The number y of shares traded (in millions) on the New York Stock Exchange from 1990 through 2002 can be modeled by

$$y = 36{,}663e^{0.1902t}$$

where t represents the year, with $t = 0$ corresponding to 1990. At what rate was the number of shares traded changing in 1998? *(Source: New York Stock Exchange, Inc.)*

Solution The derivative of the given model is

$$y' = (0.1902)(36{,}663)e^{0.1902t}$$

$$\approx 6973e^{0.1902t}.$$

By evaluating the derivative when $t = 8$, you can conclude that the rate of change in 1998 was about

31,933 million shares per year.

The graph of this model is shown in Figure 5.23.

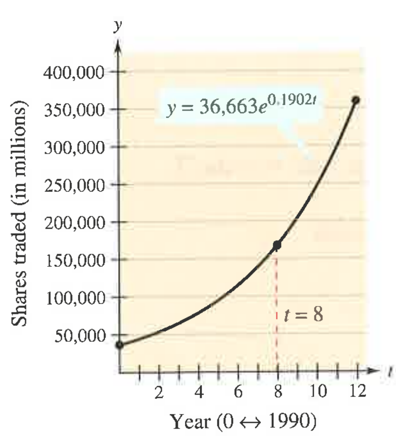


Figure 5.23

Integrals of Exponential Functions

Each differentiation formula in Theorem 5.11 has a corresponding integration formula.

THEOREM 5.12 Integration Rules for Exponential Functions

Let u be a differentiable function of x.

1. $\displaystyle\int e^x\,dx = e^x + C$ **2.** $\displaystyle\int e^u\,du = e^u + C$

EXAMPLE 7 Integrating Exponential Functions

Find $\displaystyle\int e^{3x+1}\,dx.$

Solution If you let $u = 3x + 1$, then $du = 3\,dx$.

$$\int e^{3x+1}dx = \frac{1}{3}\int e^{3x+1}(3)\,dx \qquad \text{Multiply and divide by 3.}$$

$$= \frac{1}{3}\int e^u\,du \qquad \text{Substitute: } u = 3x + 1.$$

$$= \frac{1}{3}e^u + C \qquad \text{Apply Exponential Rule.}$$

$$= \frac{e^{3x+1}}{3} + C \qquad \text{Back-substitute.}$$

NOTE In Example 7, the missing *constant* factor 3 was introduced to create $du = 3\,dx$. However, remember that you cannot introduce a missing *variable* factor in the integrand. For instance,

$$\int e^{-x^2}\,dx \neq \frac{1}{x}\int e^{-x^2}(x\,dx).$$

EXAMPLE 8 Integrating Exponential Functions

Find $\displaystyle\int 5xe^{-x^2}\,dx.$

Solution If you let $u = -x^2$, then $du = -2x\,dx$ or $x\,dx = -du/2$.

$$\int 5xe^{-x^2}dx = \int 5e^{-x^2}(x\,dx) \qquad \text{Regroup integrand.}$$

$$= \int 5e^u\left(-\frac{du}{2}\right) \qquad \text{Substitute: } u = -x^2.$$

$$= -\frac{5}{2}\int e^u\,du \qquad \text{Constant Multiple Rule}$$

$$= -\frac{5}{2}e^u + C \qquad \text{Apply Exponential Rule.}$$

$$= -\frac{5}{2}e^{-x^2} + C \qquad \text{Back-substitute.}$$

EXAMPLE 9 Integrating Exponential Functions

a. $$\int \frac{e^{1/x}}{x^2}\,dx = -\int \overbrace{e^{1/x}}^{e^u}\overbrace{\left(-\frac{1}{x^2}\right) dx}^{du} \qquad u = \frac{1}{x}$$
$$= -e^{1/x} + C$$

b. $$\int \sin x\, e^{\cos x}\,dx = -\int \overbrace{e^{\cos x}}^{e^u}\overbrace{(-\sin x\,dx)}^{du} \qquad u = \cos x$$
$$= -e^{\cos x} + C$$

EXAMPLE 10 Finding Areas Bounded by Exponential Functions

Evaluate each definite integral.

a. $\displaystyle\int_0^1 e^{-x}\,dx$ **b.** $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx$ **c.** $\displaystyle\int_{-1}^0 [e^x\cos(e^x)]\,dx$

Solution

a. $$\int_0^1 e^{-x}\,dx = -e^{-x}\Big]_0^1 \qquad \text{See Figure 5.24(a).}$$
$$= -e^{-1} - (-1)$$
$$= 1 - \frac{1}{e}$$
$$\approx 0.632$$

b. $$\int_0^1 \frac{e^x}{1+e^x}\,dx = \ln(1+e^x)\Big]_0^1 \qquad \text{See Figure 5.24(b).}$$
$$= \ln(1+e) - \ln 2$$
$$\approx 0.620$$

c. $$\int_{-1}^0 [e^x\cos(e^x)]\,dx = \sin(e^x)\Big]_{-1}^0 \qquad \text{See Figure 5.24(c).}$$
$$= \sin 1 - \sin(e^{-1})$$
$$\approx 0.482$$

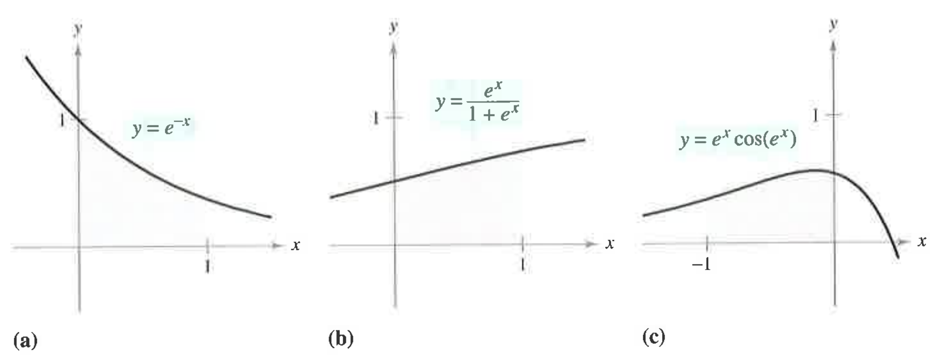


Figure 5.24

Exercises for Section 5.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–14, solve for x accurate to three decimal places.

1. $e^{\ln x} = 4$ **2.** $e^{\ln 2x} = 12$

3. $e^x = 12$ **4.** $4e^x = 83$

5. $9 - 2e^x = 7$ **6.** $-6 + 3e^x = 8$

7. $50e^{-x} = 30$ **8.** $200e^{-4x} = 15$

9. $\ln x = 2$ **10.** $\ln x^2 = 10$

11. $\ln(x - 3) = 2$ **12.** $\ln 4x = 1$

13. $\ln\sqrt{x + 2} = 1$ **14.** $\ln(x - 2)^2 = 12$

In Exercises 15–18, sketch the graph of the function.

15. $y = e^{-x}$ **16.** $y = \frac{1}{2}e^x$

17. $y = e^{-x^2}$ **18.** $y = e^{-x/2}$

19. Use a graphing utility to graph $f(x) = e^x$ and the given function in the same viewing window. How are the two graphs related?

(a) $g(x) = e^{x-2}$ (b) $h(x) = -\frac{1}{2}e^x$ (c) $q(x) = e^{-x} + 3$

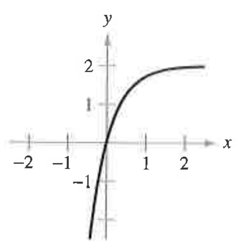
20. Use a graphing utility to graph the function. Use the graph to determine any asymptotes of the function.

(a) $f(x) = \dfrac{8}{1 + e^{-0.5x}}$

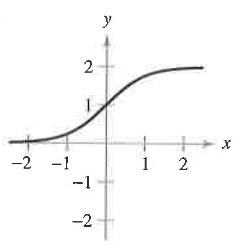
(b) $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

In Exercises 21–24, match the equation with the correct graph. Assume that a and C are positive real numbers. [The graphs are labeled (a), (b), (c), and (d).]

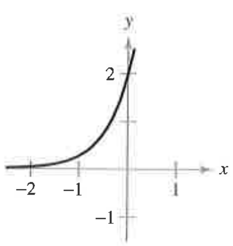
(a)



(b)



(c)



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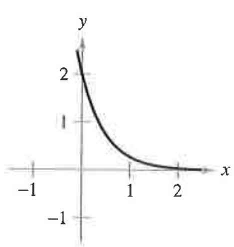


21. $y = Ce^{ax}$ **22.** $y = Ce^{-ax}$

23. $y = C(1 - e^{-ax})$ **24.** $y = \dfrac{C}{1 + e^{-ax}}$

In Exercises 25–28, illustrate that the functions are inverses of each other by graphing both functions on the same set of coordinate axes.

25. $f(x) = e^{2x}$, $g(x) = \ln\sqrt{x}$ **26.** $f(x) = e^{x/3}$, $g(x) = \ln x^3$

27. $f(x) = e^x - 1$, $g(x) = \ln(x + 1)$ **28.** $f(x) = e^{x-1}$, $g(x) = 1 + \ln x$

29. ***Graphical Analysis*** Use a graphing utility to graph

$$f(x) = \left(1 + \frac{0.5}{x}\right)^x \quad \text{and} \quad g(x) = e^{0.5}$$

in the same viewing window. What is the relationship between f and g as $x \to \infty$?

30. ***Conjecture*** Use the result of Exercise 29 to make a conjecture about the value of

$$\left(1 + \frac{r}{x}\right)^x$$

as $x \to \infty$.

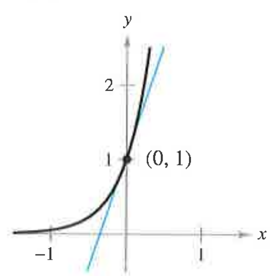
In Exercises 31 and 32, compare the given number with the number e. Is the number less than or greater than e?

31. $\left(1 + \dfrac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$ (See Exercise 30.)

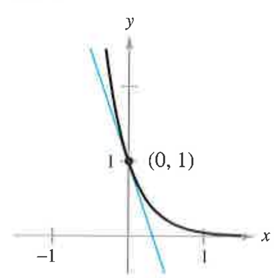
32. $1 + 1 + \dfrac{1}{2} + \dfrac{1}{6} + \dfrac{1}{24} + \dfrac{1}{120} + \dfrac{1}{720} + \dfrac{1}{5040}$

In Exercises 33 and 34, find an equation of the tangent line to the graph of the function at the point (0, 1).

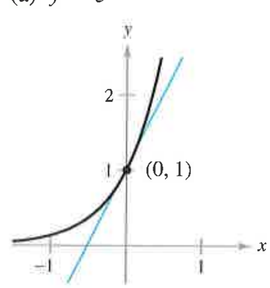
33. (a) $y = e^{3x}$



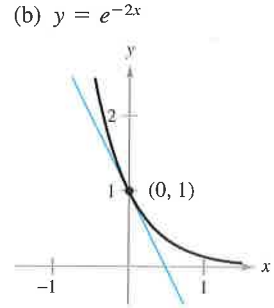
(b) $y = e^{-3x}$



34. (a) $y = e^{2x}$



(b) $y = e^{-2x}$



In Exercises 35–48, find the derivative.

35. $f(x) = e^{2x}$
36. $y = e^{-x^2}$
37. $y = e^{\sqrt{x}}$
38. $y = x^2e^{-x}$
39. $g(t) = (e^{-t} + e^t)^3$
40. $g(t) = e^{-3/t^2}$
41. $y = \ln(1 + e^{2x})$
42. $y = \ln\left(\dfrac{1 + e^x}{1 - e^x}\right)$
43. $y = \dfrac{2}{e^x + e^{-x}}$
44. $y = \dfrac{e^x - e^{-x}}{2}$
45. $y = e^x(\sin x + \cos x)$
46. $y = \ln e^x$
47. $F(x) = \displaystyle\int_{\pi}^{\ln x} \cos e^t\,dt$
48. $F(x) = \displaystyle\int_0^{e^{2x}} \ln(t + 1)\,dt$

In Exercises 49–56, find an equation of the tangent line to the graph of the function at the given point.

49. $f(x) = e^{1-x}, \quad (1, 1)$
50. $y = e^{-2x+x^2}, \quad (2, 1)$
51. $y = \ln(e^{x^2}), \quad (-2, 4)$
52. $y = \ln\dfrac{e^x + e^{-x}}{2}, \quad (0, 0)$
53. $y = x^2e^x - 2xe^x + 2e^x, \quad (1, e)$
54. $y = xe^x - e^x, \quad (1, 0)$
55. $f(x) = e^{-x}\ln x, \quad (1, 0)$
56. $f(x) = e^3 \ln x, \quad (1, 0)$

In Exercises 57 and 58, use implicit differentiation to find dy/dx.

57. $xe^y - 10x + 3y = 0$
58. $e^{xy} + x^2 - y^2 = 10$

In Exercises 59 and 60, find an equation of the tangent line to the graph of the function at the given point.

59. $xe^y + ye^x = 1, \quad (0, 1)$
60. $1 + \ln xy = e^{x-y}, \quad (1, 1)$

In Exercises 61 and 62, find the second derivative of the function.

61. $f(x) = (3 + 2x)e^{-3x}$
62. $g(x) = \sqrt{x} + e^x \ln x$

In Exercises 63 and 64, show that the function $y = f(x)$ is a solution of the differential equation.

63. $y = e^x\left(\cos\sqrt{2}x + \sin\sqrt{2}x\right)$
$y'' - 2y' + 3y = 0$
64. $y = e^x(3\cos 2x - 4\sin 2x)$
$y'' - 2y' + 5y = 0$

In Exercises 65–72, find the extrema and the points of inflection (if any exist) of the function. Use a graphing utility to graph the function and confirm your results.

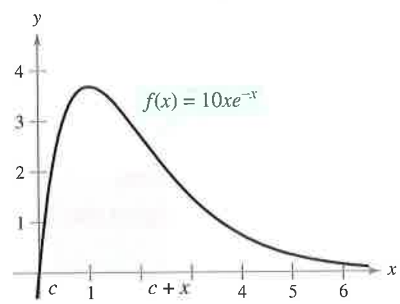
65. $f(x) = \dfrac{e^x + e^{-x}}{2}$
66. $f(x) = \dfrac{e^x - e^{-x}}{2}$
67. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-2)^2/2}$
68. $g(x) = \dfrac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}$
69. $f(x) = x^2e^{-x}$
70. $f(x) = xe^{-x}$
71. $g(t) = 1 + (2 + t)e^{-t}$
72. $f(x) = -2 + e^{3x}(4 - 2x)$

73. ***Area*** Find the area of the largest rectangle that can be inscribed under the curve $y = e^{-x^2}$ in the first and second quadrants.

74. ***Area*** Perform the following steps to find the maximum area of the rectangle shown in the figure.
(a) Solve for c in the equation $f(c) = f(c + x)$.
(b) Use the result in part (a) to write the area A as a function of x. [*Hint*: $A = xf(c)$]
(c) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions of the rectangle of maximum area. Determine the maximum area.
(d) Use a graphing utility to graph the expression for c found in part (a). Use the graph to approximate

$$\lim_{x\to 0^+} c \quad \text{and} \quad \lim_{x\to\infty} c.$$

Use this result to describe the changes in dimensions and position of the rectangle for $0 < x < \infty$.



75. Verify that the function

$$y = \frac{L}{1 + ae^{-x/b}}, \quad a > 0, \quad b > 0, \quad L > 0$$

increases at a maximum rate when $y = L/2$.

76. ***Writing*** Consider the function $f(x) = \dfrac{2}{1 + e^{1/x}}$.
(a) Use a graphing utility to graph f.
(b) Write a short paragraph explaining why the graph has a horizontal asymptote at $y = 1$ and why the function has a nonremovable discontinuity at $x = 0$.

77. Find a point on the graph of the function $f(x) = e^{2x}$ such that the tangent line to the graph at that point passes through the origin. Use a graphing utility to graph f and the tangent line in the same viewing window.

78. Find the point on the graph of $y = e^{-x}$ where the normal line to the curve passes through the origin. (Use Newton's Method or the *zero* or *root* feature of a graphing utility.)

79. ***Depreciation*** The value V of an item t years after it is purchased is $V = 15{,}000e^{-0.6286t}$, $0 \le t \le 10$.
(a) Use a graphing utility to graph the function.
(b) Find the rate of change of V with respect to t when $t = 1$ and $t = 5$.
(c) Use a graphing utility to graph the tangent line to the function when $t = 1$ and $t = 5$.

80. *Harmonic Motion* The displacement from equilibrium of a mass oscillating on the end of a spring suspended from a ceiling is

$$y = 1.56e^{-0.22t} \cos 4.9t$$

where y is the displacement in feet and t is the time in seconds. Use a graphing utility to graph the displacement function on the interval $[0, 10]$. Find a value of t past which the displacement is less than 3 inches from equilibrium.

81. *Modeling Data* A meteorologist measures the atmospheric pressure P (in kilograms per square meter) at altitude h (in kilometers). The data are shown below.

h	0	5	10	15	20
P	10,332	5583	2376	1240	517

(a) Use a graphing utility to plot the points $(h, \ln P)$. Use the regression capabilities of the graphing utility to find a linear model for the revised data points.

(b) The line in part (a) has the form

$$\ln P = ah + b.$$

Write the equation in exponential form.

(c) Use a graphing utility to plot the original data and graph the exponential model in part (b).

(d) Find the rate of change of the pressure when $h = 5$ and $h = 18$.

82. *Modeling Data* The table lists the approximate value V of a mid-sized sedan for the years 1997 through 2003. The variable t represents the time in years, with $t = 7$ corresponding to 1997.

t	7	8	9	10
V	\$17,040	\$14,590	\$12,845	\$10,995

t	11	12	13
V	\$9220	\$8095	\$6835

(a) Use a computer algebra system to find linear and quadratic models for the data. Plot the data and graph the models.

(b) What does the slope represent in the linear model in part (a)?

(c) Use a computer algebra system to fit an exponential model to the data.

(d) Determine the horizontal asymptote of the exponential model found in part (c). Interpret its meaning in the context of the problem.

(e) Find the rate of decrease in the value of the sedan when $t = 8$ and $t = 12$ using the exponential model.

Linear and Quadratic Approximations **In Exercises 83 and 84, use a graphing utility to graph the function. Then graph**

$$P_1(x) = f(0) + f'(0)(x - 0)$$

and

$$P_2(x) = f(0) + f'(0)(x - 0) + \tfrac{1}{2}f''(0)(x - 0)^2$$

in the same viewing window. Compare the values of f, P_1, and P_2 and their first derivatives at $x = 0$.

83. $f(x) = e^{x/2}$ **84.** $f(x) = e^{-x^2/2}$

In Exercises 85–98, find the indefinite integral.

85. $\int e^{5x}(5)\,dx$ **86.** $\int e^{-x^4}(-4x^3)\,dx$

87. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ **88.** $\int \frac{e^{1/x^2}}{x^3}\,dx$

89. $\int \frac{e^{-x}}{1 + e^{-x}}\,dx$ **90.** $\int \frac{e^{2x}}{1 + e^{2x}}\,dx$

91. $\int e^x\sqrt{1 - e^x}\,dx$ **92.** $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$

93. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}}\,dx$ **94.** $\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2}\,dx$

95. $\int \frac{5 - e^x}{e^{2x}}\,dx$ **96.** $\int \frac{e^{2x} + 2e^x + 1}{e^x}\,dx$

97. $\int e^{-x}\tan(e^{-x})\,dx$ **98.** $\int \ln(e^{2x-1})\,dx$

In Exercises 99–106, evaluate the definite integral. Use a graphing utility to verify your result.

99. $\int_0^1 e^{-2x}\,dx$ **100.** $\int_3^4 e^{3-x}\,dx$

101. $\int_0^1 xe^{-x^2}\,dx$ **102.** $\int_{-2}^0 x^2 e^{x^3/2}\,dx$

103. $\int_1^3 \frac{e^{3/x}}{x^2}\,dx$ **104.** $\int_0^{\sqrt{2}} xe^{-(x^2/2)}\,dx$

105. $\int_0^{\pi/2} e^{\sin \pi x}\cos \pi x\,dx$ **106.** $\int_{\pi/3}^{\pi/2} e^{\sec 2x}\sec 2x \tan 2x\,dx$

Differential Equations **In Exercises 107 and 108, solve the differential equation.**

107. $\frac{dy}{dx} = xe^{ax^2}$ **108.** $\frac{dy}{dx} = (e^x - e^{-x})^2$

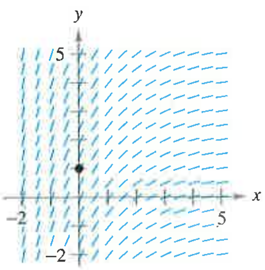
Differential Equations **In Exercises 109 and 110, find the particular solution that satisfies the initial conditions.**

109. $f''(x) = \frac{1}{2}(e^x + e^{-x})$, $f(0) = 1, f'(0) = 0$

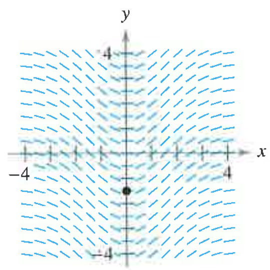
110. $f''(x) = \sin x + e^{2x}$, $f(0) = \frac{1}{4}, f'(0) = \frac{1}{2}$

Slope Fields **In Exercises 111 and 112, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

111. $\dfrac{dy}{dx} = 2e^{-x/2}$, $(0, 1)$



112. $\dfrac{dy}{dx} = xe^{-0.2x^2}$, $\left(0, -\dfrac{3}{2}\right)$



Area **In Exercises 113–116, find the area of the region bounded by the graphs of the equations. Use a graphing utility to graph the region and verify your result.**

113. $y = e^x, y = 0, x = 0, x = 5$

114. $y = e^{-x}, y = 0, x = a, x = b$

115. $y = xe^{-x^2/4}, y = 0, x = 0, x = \sqrt{6}$

116. $y = e^{-2x} + 2, y = 0, x = 0, x = 2$

Numerical Integration **In Exercises 117 and 118, approximate the integral using the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule with $n = 12$. Use a graphing utility to verify your results.**

117. $\displaystyle\int_0^4 \sqrt{x}\, e^x \, dx$

118. $\displaystyle\int_0^2 2xe^{-x} \, dx$

119. ***Probability*** A car battery has an average lifetime of 48 months with a standard deviation of 6 months. The battery lives are normally distributed. The probability that a given battery will last between 48 months and 60 months is

$$0.0665 \int_{48}^{60} e^{-0.0139(t-48)^2} \, dt.$$

Use the integration capabilities of a graphing utility to approximate the integral. Interpret the resulting probability.

120. ***Probability*** The median waiting time (in minutes) for people waiting for service in a convenience store is given by the solution of the equation

$$\int_0^x 0.3e^{-0.3t} \, dt = \frac{1}{2}.$$

Solve the equation.

121. Given $e^x \geq 1$ for $x \geq 0$, it follows that

$$\int_0^x e^t \, dt \geq \int_0^x 1 \, dt.$$

Perform this integration to derive the inequality $e^x \geq 1 + x$ for $x \geq 0$.

122. ***Modeling Data*** A valve on a storage tank is opened for 4 hours to release a chemical in a manufacturing process. The flow rate R (in liters per hour) at time t (in hours) is given in the table.

t	0	1	2	3	4
R	425	240	118	71	36

(a) Use the regression capabilities of a graphing utility to find a linear model for the points $(t, \ln R)$. Write the resulting equation of the form $\ln R = at + b$ in exponential form.

(b) Use a graphing utility to plot the data and graph the exponential model.

(c) Use the definite integral to approximate the number of liters of chemical released during the 4 hours.

Writing About Concepts

123. In your own words, state the properties of the natural exponential function.

124. Describe the relationship between the graphs of $f(x) = \ln x$ and $g(x) = e^x$.

125. Is there a function f such that $f(x) = f'(x)$? If so, identify it.

126. Without integrating, state the integration formula you can use to integrate each of the following.

(a) $\displaystyle\int \frac{e^x}{e^x + 1} \, dx$ (b) $\displaystyle\int xe^{x^2} \, dx$

127. Find, to three decimal places, the value of x such that $e^{-x} = x$. (Use Newton's Method or the *zero* or *root* feature of a graphing utility.)

128. Find the value of a such that the area bounded by $y = e^{-x}$, the x-axis, $x = -a$, and $x = a$ is $\frac{8}{3}$.

129. Prove that $\dfrac{e^a}{e^b} = e^{a-b}$.

130. Let $f(x) = \dfrac{\ln x}{x}$.

(a) Graph f on $(0, \infty)$ and show that f is strictly decreasing on (e, ∞).

(b) Show that if $e \leq A < B$, then $A^B > B^A$.

(c) Use part (b) to show that $e^\pi > \pi^e$.

Section 5.5

Bases Other Than e and Applications

- **Define exponential functions that have bases other than *e*.**
- **Differentiate and integrate exponential functions that have bases other than *e*.**
- **Use exponential functions to model compound interest and exponential growth.**

Bases Other than e

The **base** of the natural exponential function is e. This "natural" base can be used to assign a meaning to a general base a.

Definition of Exponential Function to Base *a*

If a is a positive real number $(a \neq 1)$ and x is any real number, then the **exponential function to the base *a*** is denoted by a^x and is defined by

$$a^x = e^{(\ln a)x}.$$

If $a = 1$, then $y = 1^x = 1$ is a constant function.

These functions obey the usual laws of exponents. For instance, here are some familiar properties.

1. $a^0 = 1$ **2.** $a^x a^y = a^{x+y}$

3. $\dfrac{a^x}{a^y} = a^{x-y}$ **4.** $(a^x)^y = a^{xy}$

When modeling the half-life of a radioactive sample, it is convenient to use $\frac{1}{2}$ as the base of the exponential model.

EXAMPLE 1 Radioactive Half-Life Model

The half-life of carbon-14 is about 5715 years. A sample contains 1 gram of carbon-14. How much will be present in 10,000 years?

Solution Let $t = 0$ represent the present time and let y represent the amount (in grams) of carbon-14 in the sample. Using a base of $\frac{1}{2}$, you can model y by the equation

$$y = \left(\frac{1}{2}\right)^{t/5715}.$$

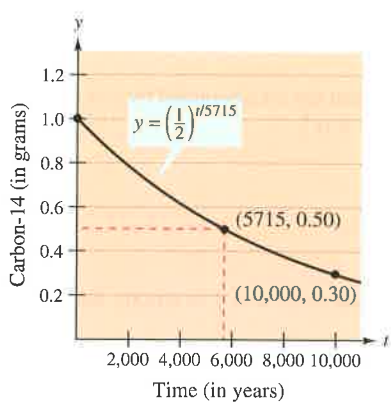
Notice that when $t = 5715$, the amount is reduced to half of the original amount.

$$y = \left(\frac{1}{2}\right)^{5715/5715} = \frac{1}{2} \text{ gram}$$

When $t = 11{,}430$, the amount is reduced to a quarter of the original amount, and so on. To find the amount of carbon-14 after 10,000 years, substitute 10,000 for t.

$$y = \left(\frac{1}{2}\right)^{10{,}000/5715} \approx 0.30 \text{ gram}$$

The graph of y is shown in Figure 5.25.



The half-life of carbon-14 is about 5715 years.
Figure 5.25

Logarithmic functions to bases other than e can be defined in much the same way as exponential functions to other bases are defined.

NOTE In precalculus, you learned that $\log_a x$ is the value to which a must be raised to produce x. This agrees with the definition given here because

$$\begin{aligned} a^{\log_a x} &= a^{(1/\ln a)\ln x} \\ &= (e^{\ln a})^{(1/\ln a)\ln x} \\ &= e^{(\ln a/\ln a)\ln x} \\ &= e^{\ln x} \\ &= x. \end{aligned}$$

Definition of Logarithmic Function to Base *a*

If a is a positive real number $(a \neq 1)$ and x is any positive real number, then the **logarithmic function to the base *a*** is denoted by $\log_a x$ and is defined as

$$\log_a x = \frac{1}{\ln a} \ln x.$$

Logarithmic functions to the base a have properties similar to those of the natural logarithmic function given in Theorem 5.2.

1. $\log_a 1 = 0$ — Log of 1

2. $\log_a xy = \log_a x + \log_a y$ — Log of a product

3. $\log_a x^n = n \log_a x$ — Log of a power

4. $\log_a \dfrac{x}{y} = \log_a x - \log_a y$ — Log of a quotient

From the definitions of the exponential and logarithmic functions to the base a, it follows that $f(x) = a^x$ and $g(x) = \log_a x$ are inverse functions of each other.

Properties of Inverse Functions

1. $y = a^x$ if and only if $x = \log_a y$

2. $a^{\log_a x} = x$, for $x > 0$

3. $\log_a a^x = x$, for all x

The logarithmic function to the base 10 is called the **common logarithmic function.** So, for common logarithms, $y = 10^x$ if and only if $x = \log_{10} y$.

EXAMPLE 2 Bases Other Than e

Solve for x in each equation.

a. $3^x = \dfrac{1}{81}$

b. $\log_2 x = -4$

Solution

a. To solve this equation, you can apply the logarithmic function to the base 3 to each side of the equation.

$$\begin{aligned} 3^x &= \frac{1}{81} \\ \log_3 3^x &= \log_3 \frac{1}{81} \\ x &= \log_3 3^{-4} \\ x &= -4 \end{aligned}$$

b. To solve this equation, you can apply the exponential function to the base 2 to each side of the equation.

$$\begin{aligned} \log_2 x &= -4 \\ 2^{\log_2 x} &= 2^{-4} \\ x &= \frac{1}{2^4} \\ x &= \frac{1}{16} \end{aligned}$$

Differentiation and Integration

To differentiate exponential and logarithmic functions to other bases, you have three options: (1) use the definitions of a^x and $\log_a x$ and differentiate using the rules for the natural exponential and logarithmic functions, (2) use logarithmic differentiation, or (3) use the following differentiation rules for bases other than e.

THEOREM 5.13 Derivatives for Bases Other Than e

Let a be a positive real number ($a \neq 1$) and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[a^x] = (\ln a)a^x$
2. $\dfrac{d}{dx}[a^u] = (\ln a)a^u\dfrac{du}{dx}$
3. $\dfrac{d}{dx}[\log_a x] = \dfrac{1}{(\ln a)x}$
4. $\dfrac{d}{dx}[\log_a u] = \dfrac{1}{(\ln a)u}\dfrac{du}{dx}$

Proof By definition, $a^x = e^{(\ln a)x}$. So, you can prove the first rule by letting $u = (\ln a)x$ and differentiating with base e to obtain

$$\frac{d}{dx}[a^x] = \frac{d}{dx}[e^{(\ln a)x}] = e^u\frac{du}{dx} = e^{(\ln a)x}(\ln a) = (\ln a)a^x.$$

To prove the third rule, you can write

$$\frac{d}{dx}[\log_a x] = \frac{d}{dx}\left[\frac{1}{\ln a}\ln x\right] = \frac{1}{\ln a}\left(\frac{1}{x}\right) = \frac{1}{(\ln a)x}.$$

The second and fourth rules are simply the Chain Rule versions of the first and third rules.

NOTE These differentiation rules are similar to those for the natural exponential function and natural logarithmic function. In fact, they differ only by the constant factors $\ln a$ and $1/\ln a$. This points out one reason why, for calculus, e is the most convenient base.

EXAMPLE 3 Differentiating Functions to Other Bases

Find the derivative of each function.

a. $y = 2^x$

b. $y = 2^{3x}$

c. $y = \log_{10} \cos x$

Solution

a. $y' = \dfrac{d}{dx}[2^x] = (\ln 2)2^x$

b. $y' = \dfrac{d}{dx}[2^{3x}] = (\ln 2)2^{3x}(3) = (3 \ln 2)2^{3x}$

Try writing 2^{3x} as 8^x and differentiating to see that you obtain the same result.

c. $y' = \dfrac{d}{dx}[\log_{10} \cos x] = \dfrac{-\sin x}{(\ln 10)\cos x} = -\dfrac{1}{\ln 10}\tan x$

Occasionally, an integrand involves an exponential function to a base other than e. When this occurs, there are two options: (1) convert to base e using the formula $a^x = e^{(\ln a)x}$ and then integrate, or (2) integrate directly, using the integration formula

$$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$$

(which follows from Theorem 5.13).

EXAMPLE 4 Integrating an Exponential Function to Another Base

Find $\int 2^x\,dx$.

Solution

$$\int 2^x\,dx = \frac{1}{\ln 2}2^x + C$$

When the Power Rule, $D_x[x^n] = nx^{n-1}$, was introduced in Chapter 2, the exponent n was required to be a rational number. Now the rule is extended to cover any real value of n. Try to prove this theorem using logarithmic differentiation.

THEOREM 5.14 The Power Rule for Real Exponents

Let n be any real number and let u be a differentiable function of x.

1. $\dfrac{d}{dx}[x^n] = nx^{n-1}$
2. $\dfrac{d}{dx}[u^n] = nu^{n-1}\dfrac{du}{dx}$

The next example compares the derivatives of four types of functions. Each function uses a different differentiation formula, depending on whether the base and exponent are constants or variables.

EXAMPLE 5 Comparing Variables and Constants

a. $\dfrac{d}{dx}[e^e] = 0$ Constant Rule

b. $\dfrac{d}{dx}[e^x] = e^x$ Exponential Rule

c. $\dfrac{d}{dx}[x^e] = ex^{e-1}$ Power Rule

d. $y = x^x$ Logarithmic differentiation

$$\ln y = \ln x^x$$
$$\ln y = x \ln x$$
$$\frac{y'}{y} = x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$$
$$y' = y(1 + \ln x) = x^x(1 + \ln x)$$

NOTE Be sure you see that there is no simple differentiation rule for calculating the derivative of $y = x^x$. In general, if $y = u(x)^{v(x)}$, you need to use logarithmic differentiation.

Exercises for Section 5.5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, evaluate the expression without using a calculator.

1. $\log_2 \frac{1}{8}$

2. $\log_{27} 9$

3. $\log_7 1$

4. $\log_a \dfrac{1}{a}$

In Exercises 5–8, write the exponential equation as a logarithmic equation or vice versa.

5. (a) $2^3 = 8$
(b) $3^{-1} = \frac{1}{3}$

6. (a) $27^{2/3} = 9$
(b) $16^{3/4} = 8$

7. (a) $\log_{10} 0.01 = -2$
(b) $\log_{0.5} 8 = -3$

8. (a) $\log_3 \frac{1}{9} = -2$
(b) $49^{1/2} = 7$

In Exercises 9–14, sketch the graph of the function by hand.

9. $y = 3^x$

10. $y = 3^{x-1}$

11. $y = \left(\frac{1}{3}\right)^x$

12. $y = 2^{x^2}$

13. $h(x) = 5^{x-2}$

14. $y = 3^{-|x|}$

In Exercises 15–20, solve for *x* or *b*.

15. (a) $\log_{10} 1000 = x$
(b) $\log_{10} 0.1 = x$

16. (a) $\log_3 \frac{1}{81} = x$
(b) $\log_6 36 = x$

17. (a) $\log_3 x = -1$
(b) $\log_2 x = -4$

18. (a) $\log_b 27 = 3$
(b) $\log_b 125 = 3$

19. (a) $x^2 - x = \log_5 25$
(b) $3x + 5 = \log_2 64$

20. (a) $\log_3 x + \log_3(x - 2) = 1$
(b) $\log_{10}(x + 3) - \log_{10} x = 1$

In Exercises 21–30, solve the equation accurate to three decimal places.

21. $3^{2x} = 75$

22. $5^{6x} = 8320$

23. $2^{3-z} = 625$

24. $3(5^{x-1}) = 86$

25. $\left(1 + \dfrac{0.09}{12}\right)^{12t} = 3$

26. $\left(1 + \dfrac{0.10}{365}\right)^{365t} = 2$

27. $\log_2(x - 1) = 5$

28. $\log_{10}(t - 3) = 2.6$

29. $\log_3 x^2 = 4.5$

30. $\log_5 \sqrt{x - 4} = 3.2$

In Exercises 31–34, use a graphing utility to graph the function and approximate its zero(s) accurate to three decimal places.

31. $g(x) = 6(2^{1-x}) - 25$

32. $f(t) = 300(1.0075^{12t}) - 735.41$

33. $h(s) = 32 \log_{10}(s - 2) + 15$

34. $g(x) = 1 - 2 \log_{10}[x(x - 3)]$

In Exercises 35 and 36, illustrate that the functions are inverse functions of each other by sketching their graphs on the same set of coordinate axes.

35. $f(x) = 4^x$
$g(x) = \log_4 x$

36. $f(x) = 3^x$
$g(x) = \log_3 x$

In Exercises 37–48, find the derivative of the function.

37. $f(x) = 4^x$

38. $y = x(6^{-2x})$

39. $g(t) = t^2 2^t$

40. $f(t) = \dfrac{3^{2t}}{t}$

41. $h(\theta) = 2^{-\theta} \cos \pi\theta$

42. $g(\alpha) = 5^{-\alpha/2} \sin 2\alpha$

43. $f(x) = \log_2 \dfrac{x^2}{x - 1}$

44. $h(x) = \log_3 \dfrac{x\sqrt{x - 1}}{2}$

45. $y = \log_5 \sqrt{x^2 - 1}$

46. $y = \log_{10} \dfrac{x^2 - 1}{x}$

47. $g(t) = \dfrac{10 \log_4 t}{t}$

48. $f(t) = t^{3/2} \log_2 \sqrt{t + 1}$

In Exercises 49–52, find an equation of the tangent line to the graph of the function at the given point.

49. $y = 2^{-x}$, $(-1, 2)$

50. $y = 5^{x-2}$, $(2, 1)$

51. $y = \log_3 x$, $(27, 3)$

52. $y = \log_{10} 2x$, $(5, 1)$

In Exercises 53–56, use logarithmic differentiation to find *dy/dx*.

53. $y = x^{2/x}$

54. $y = x^{x-1}$

55. $y = (x - 2)^{x+1}$

56. $y = (1 + x)^{1/x}$

In Exercises 57–60, find an equation of the tangent line to the graph of the function at the given point.

57. $y = x^{\sin x}$, $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

58. $y = (\sin x)^{2x}$, $\left(\dfrac{\pi}{2}, 1\right)$

59. $y = (\ln x)^{\cos x}$, $(e, 1)$

60. $y = x^{1/x}$, $(1, 1)$

In Exercises 61–66, find the integral.

61. $\int 3^x\,dx$

62. $\int 5^{-x}\,dx$

63. $\int x(5^{-x^2})\,dx$

64. $\int (3-x)7^{(3-x)^2}\,dx$

65. $\int \frac{3^{2x}}{1+3^{2x}}\,dx$

66. $\int 2^{\sin x}\cos x\,dx$

In Exercises 67–70, evaluate the integral.

67. $\int_{-1}^{2} 2^x\,dx$

68. $\int_{-2}^{2} 4^{x/2}\,dx$

69. $\int_{0}^{1} (5^x - 3^x)\,dx$

70. $\int_{1}^{e} (6^x - 2^x)\,dx$

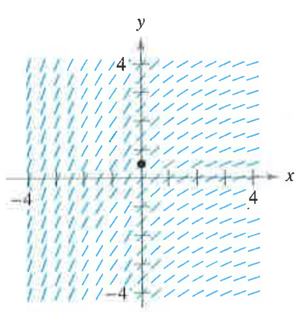
Area **In Exercises 71 and 72, find the area of the region bounded by the graphs of the equations.**

71. $y = 3^x, y = 0, x = 0, x = 3$

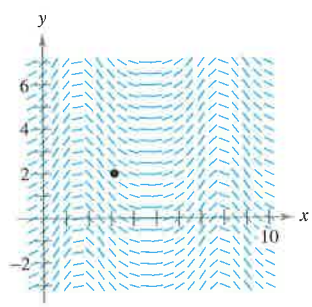
72. $y = 3^{\cos x}\sin x, y = 0, x = 0, x = \pi$

Slope Fields **In Exercises 73 and 74, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

73. $\frac{dy}{dx} = 0.4^{x/3}$, $\left(0, \frac{1}{2}\right)$



74. $\frac{dy}{dx} = e^{\sin x}\cos x$, $(\pi, 2)$



Writing About Concepts

75. The table of values below was obtained by evaluating a function. Determine which of the statements may be true and which must be false, and explain why.

(a) y is an exponential function of x.

(b) y is a logarithmic function of x.

(c) x is an exponential function of y.

(d) y is a linear function of x.

x	1	2	8
y	0	1	3

76. Consider the function $f(x) = \log_{10} x$.

(a) What is the domain of f?

(b) Find f^{-1}.

(c) If x is a real number between 1000 and 10,000, determine the interval in which $f(x)$ will be found.

(d) Determine the interval in which x will be found if $f(x)$ is negative.

(e) If $f(x)$ is increased by one unit, x must have been increased by what factor?

(f) Find the ratio of x_1 to x_2 given that $f(x_1) = 3n$ and $f(x_2) = n$.

77. Order the functions

$$f(x) = \log_2 x,\ g(x) = x^x,\ h(x) = x^2,\ \text{and}\ k(x) = 2^x$$

from the one with the greatest rate of growth to the one with the smallest rate of growth for large values of x.

78. Find the derivative of each function, given that a is constant.

(a) $y = x^a$ (b) $y = a^x$

(c) $y = x^x$ (d) $y = a^a$

79. ***Inflation*** If the annual rate of inflation averages 5% over the next 10 years, the approximate cost C of goods or services during any year in that decade is

$$C(t) = P(1.05)^t$$

where t is the time in years and P is the present cost.

(a) The price of an oil change for your car is presently \$24.95. Estimate the price 10 years from now.

(b) Find the rates of change of C with respect to t when $t = 1$ and $t = 8$.

(c) Verify that the rate of change of C is proportional to C. What is the constant of proportionality?

80. *Depreciation* After t years, the value of a car purchased for \$20,000 is

$$V(t) = 20{,}000\left(\tfrac{3}{4}\right)^t.$$

(a) Use a graphing utility to graph the function and determine the value of the car 2 years after it was purchased.

(b) Find the rates of change of V with respect to t when $t = 1$ and $t = 4$.

(c) Use a graphing utility to graph $V'(t)$ and determine the horizontal asymptote of $V'(t)$. Interpret its meaning in the context of the problem.

Compound Interest **In Exercises 81–84, complete the table to determine the balance A for P dollars invested at rate r for t years and compounded n times per year.**

n	1	2	4	12	365	Continuous compounding
A						

81. $P = \$1000$

$r = 3\frac{1}{2}\%$

$t = 10$ years

82. $P = \$2500$

$r = 6\%$

$t = 20$ years

83. $P = \$1000$

$r = 5\%$

$t = 30$ years

84. $P = \$5000$

$r = 7\%$

$t = 25$ years

Compound Interest **In Exercises 85–88, complete the table to determine the amount of money P (present value) that should be invested at rate r to produce a balance of \$100,000 in t years.**

t	1	10	20	30	40	50
P						

85. $r = 5\%$

Compounded continuously

86. $r = 6\%$

Compounded continuously

87. $r = 5\%$

Compounded monthly

88. $r = 7\%$

Compounded daily

89. *Compound Interest* Assume that you can earn 6% on an investment, compounded daily. Which of the following options would yield the greatest balance after 8 years?

(a) \$20,000 now

(b) \$30,000 after 8 years

(c) \$8000 now and \$20,000 after 4 years

(d) \$9000 now, \$9000 after 4 years, and \$9000 after 8 years

90. *Compound Interest* Consider a deposit of \$100 placed in an account for 20 years at $r\%$ compounded continuously. Use a graphing utility to graph the exponential functions giving the growth of the investment over the 20 years for each of the following interest rates. Compare the ending balances for each of the rates.

(a) $r = 3\%$

(b) $r = 5\%$

(c) $r = 6\%$

91. *Timber Yield* The yield V (in millions of cubic feet per acre) for a stand of timber at age t is

$$V = 6.7e^{(-48.1)/t}$$

where t is measured in years.

(a) Find the limiting volume of wood per acre as t approaches infinity.

(b) Find the rates at which the yield is changing when $t = 20$ years and $t = 60$ years.

92. *Learning Theory* In a group project in learning theory, a mathematical model for the proportion P of correct responses after n trials was found to be

$$P = \frac{0.86}{1 + e^{-0.25n}}.$$

(a) Find the limiting proportion of correct responses as n approaches infinity.

(b) Find the rates at which P is changing after $n = 3$ trials and $n = 10$ trials.

93. *Forest Defoliation* To estimate the amount of defoliation caused by the gypsy moth during a year, a forester counts the number of egg masses on $\frac{1}{40}$ of an acre the preceding fall. The percent of defoliation y is approximated by

$$y = \frac{300}{3 + 17e^{-0.0625x}}$$

where x is the number of egg masses in thousands. *(Source: USDA Forest Service)*

(a) Use a graphing utility to graph the function.

(b) Estimate the percent of defoliation if 2000 egg masses are counted.

(c) Estimate the number of egg masses that existed if you observe that approximately $\frac{2}{3}$ of a forest is defoliated.

(d) Use calculus to estimate the value of x for which y is increasing most rapidly.

94. ***Population Growth*** A lake is stocked with 500 fish, and their population increases according to the logistic curve

$$p(t) = \frac{10{,}000}{1 + 19e^{-t/5}}$$

where t is measured in months.

(a) Use a graphing utility to graph the function.

(b) What is the limiting size of the fish population?

(c) At what rates is the fish population changing at the end of 1 month and at the end of 10 months?

(d) After how many months is the population increasing most rapidly?

95. ***Modeling Data*** The breaking strengths B (in tons) of a steel cable of various diameters d (in inches) are shown in the table.

d	0.50	0.75	1.00	1.25	1.50	1.75
B	9.85	21.8	38.3	59.2	84.4	114.0

(a) Use the regression capabilities of a graphing utility to fit an exponential model to the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Find the rates of growth of the model when $d = 0.8$ and $d = 1.5$.

96. ***Comparing Models*** The amounts y (in billions of dollars) given to philanthropy (from individuals, foundations, corporations, and charitable bequests) in the United States for the years 1995 through 2002 are shown in the table, with $x = 5$ corresponding to 1995. *(Source: AAFRC Trust for Philanthropy)*

x	5	6	7	8	9	10	11	12
y	124.0	138.6	157.1	174.8	199.0	210.9	212.0	240.9

(a) Use the regression capabilities of a graphing utility to find the following models for the data.

$y_1 = ax + b$

$y_2 = a + b \ln x$

$y_3 = ab^x$

$y_4 = ax^b$

(b) Use a graphing utility to plot the data and graph each of the models. Which model do you think best fits the data?

(c) Interpret the slope of the linear model in the context of the problem.

(d) Find the rate of change of each of the models for the year 1996. Which model is increasing at the greatest rate in 1996?

97. ***Conjecture***

(a) Use a graphing utility to approximate the integrals of the functions

$$f(t) = 4\left(\frac{3}{8}\right)^{2t/3}, \quad g(t) = 4\left(\frac{\sqrt[3]{9}}{4}\right)^{t}, \text{ and } h(t) = 4e^{-0.653886t}$$

on the interval $[0, 4]$.

(b) Use a graphing utility to graph the three functions.

(c) Use the results in parts (a) and (b) to make a conjecture about the three functions. Could you make the conjecture using only part (a)? Explain. Prove your conjecture analytically.

98. Complete the table to demonstrate that e can also be defined as $\lim_{x \to 0^+} (1 + x)^{1/x}$.

x	1	10^{-1}	10^{-2}	10^{-4}	10^{-6}
$(1 + x)^{1/x}$					

In Exercises 99 and 100, find an exponential function that fits the experimental data collected over time t.

99.

t	0	1	2	3	4
y	1200.00	720.00	432.00	259.20	155.52

100.

t	0	1	2	3	4
y	600.00	630.00	661.50	694.58	729.30

True or False? **In Exercises 101–106, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

101. $e = \dfrac{271{,}801}{99{,}900}$

102. If $f(x) = \ln x$, then $f(e^{n+1}) - f(e^n) = 1$ for any value of n.

103. The functions $f(x) = 2 + e^x$ and $g(x) = \ln(x - 2)$ are inverse functions of each other.

104. The exponential function $y = Ce^x$ is a solution of the differential equation $d^n y/dx^n = y$, $n = 1, 2, 3, \ldots$.

105. The graphs of $f(x) = e^x$ and $g(x) = e^{-x}$ meet at right angles.

106. If $f(x) = g(x)e^x$, then the only zeros of f are the zeros of g.

107. Solve the logistic differential equation

$$\frac{dy}{dt} = \frac{8}{25}y\left(\frac{5}{4} - y\right), \quad y(0) = 1$$

and obtain the logistic growth function in Example 7.

$$\left[\textit{Hint: } \frac{1}{y\left(\frac{5}{4} - y\right)} = \frac{4}{5}\left(\frac{1}{y} + \frac{1}{\frac{5}{4} - y}\right)\right]$$

108. Given the exponential function $f(x) = a^x$, show that

(a) $f(u + v) = f(u) \cdot f(v)$.

(b) $f(2x) = [f(x)]^2$.

109. (a) Determine y' given $y^x = x^y$.

(b) Find the slope of the tangent line to the graph of $y^x = x^y$ at each of the following points.

(i) (c, c)

(ii) $(2, 4)$

(iii) $(4, 2)$

(c) At what point on the graph of $y^x = x^y$ does the tangent line not exist?

110. Consider the functions $f(x) = 1 + x$ and $g(x) = b^x$, $b > 1$.

(a) Given $b = 2$, use a graphing utility to graph f and g in the same viewing window. Identify the point(s) of intersection.

(b) Repeat part (a) using $b = 3$.

(c) Find all values of b such that $g(x) \geq f(x)$ for all x.

Putnam Exam Challenge

111. Which is greater

$$\left(\sqrt{n}\right)^{\sqrt{n+1}} \quad \text{or} \quad \left(\sqrt{n+1}\right)^{\sqrt{n}}$$

where $n > 8$?

112. Show that if x is positive, then

$$\log_e\left(1 + \frac{1}{x}\right) > \frac{1}{1 + x}.$$

Section Project: Using Graphing Utilities to Estimate Slope

Let $f(x) = \begin{cases} |x|^x, & x \neq 0 \\ 1, & x = 0. \end{cases}$

(a) Use a graphing utility to graph f in the viewing window $-3 \leq x \leq 3$, $-2 \leq y \leq 2$. What is the domain of f?

(b) Use the *zoom* and *trace* features of a graphing utility to estimate

$$\lim_{x \to 0} f(x).$$

(c) Write a short paragraph explaining why the function f is continuous for all real numbers.

(d) Visually estimate the slope of f at the point $(0, 1)$.

(e) Explain why the derivative of a function can be approximated by the formula

$$\frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

for small values of Δx. Use this formula to approximate the slope of f at the point $(0, 1)$.

$$f'(0) \approx \frac{f(0 + \Delta x) - f(0 - \Delta x)}{2\Delta x} = \frac{f(\Delta x) - f(-\Delta x)}{2\Delta x}$$

What do you think the slope of the graph of f is at $(0, 1)$?

(f) Find a formula for the derivative of f and determine $f'(0)$. Write a short paragraph explaining how a graphing utility might lead you to approximate the slope of a graph incorrectly.

(g) Use your formula for the derivative of f to find the relative extrema of f. Verify your answer with a graphing utility.

FOR FURTHER INFORMATION For more information on using graphing utilities to estimate slope, see the article "Computer-Aided Delusions" by Richard L. Hall in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

Section 5.6

Inverse Trigonometric Functions: Differentiation

- **Develop properties of the six inverse trigonometric functions.**
- **Differentiate an inverse trigonometric function.**
- **Review the basic differentiation rules for elementary functions.**

Inverse Trigonometric Functions

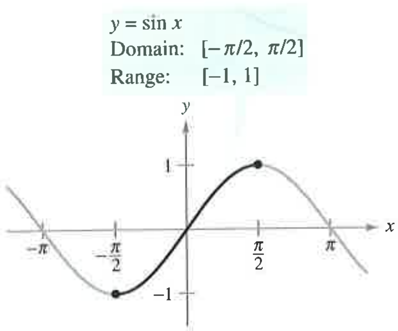
This section begins with a rather surprising statement: *None of the six basic trigonometric functions has an inverse function.* This statement is true because all six trigonometric functions are periodic and therefore are not one-to-one. In this section you will examine these six functions to see whether their domains can be redefined in such a way that they will have inverse functions on the *restricted domains.*

In Example 4 of Section 5.3, you saw that the sine function is increasing (and therefore is one-to-one) on the interval $[-\pi/2, \pi/2]$ (see Figure 5.28). On this interval you can define the inverse of the *restricted* sine function to be

$$y = \arcsin x \quad \text{if and only if} \quad \sin y = x$$

where $-1 \le x \le 1$ and $-\pi/2 \le \arcsin x \le \pi/2$.

Under suitable restrictions, each of the six trigonometric functions is one-to-one and so has an inverse function, as shown in the following definition.



The sine function is one-to-one on $[-\pi/2, \pi/2]$.
Figure 5.28

NOTE The term "iff" is used to represent the phrase "if and only if."

Definitions of Inverse Trigonometric Functions

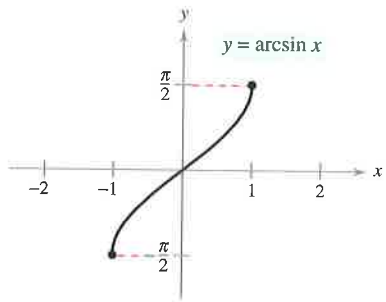
Function	*Domain*	*Range*
$y = \arcsin x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \text{arccot } x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \text{arcsec } x$ iff $\sec y = x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi,\ \ y \ne \frac{\pi}{2}$
$y = \text{arccsc } x$ iff $\csc y = x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ \ y \ne 0$

NOTE The term "arcsin x" is read as "the arcsine of x" or sometimes "the angle whose sine is x." An alternative notation for the inverse sine function is "$\sin^{-1} x$."

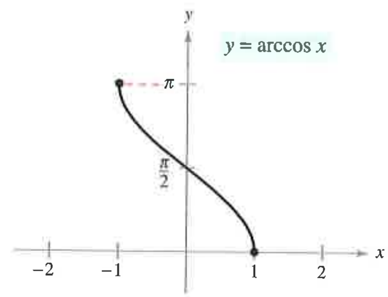
EXPLORATION

The Inverse Secant Function In the definition above, the inverse secant function is defined by restricting the domain of the secant function to the intervals $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. Most other texts and reference books agree with this, but some disagree. What other domains might make sense? Explain your reasoning graphically. Most calculators do not have a key for the inverse secant function. How can you use a calculator to evaluate the inverse secant function?

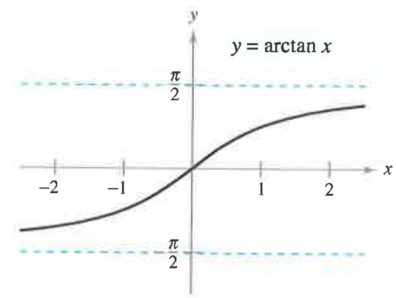
The graphs of the six inverse trigonometric functions are shown in Figure 5.29.



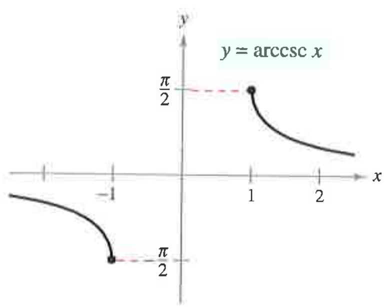
Domain: $[-1, 1]$
Range: $[-\pi/2, \pi/2]$



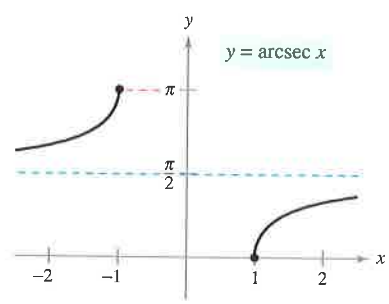
Domain: $[-1, 1]$
Range: $[0, \pi]$



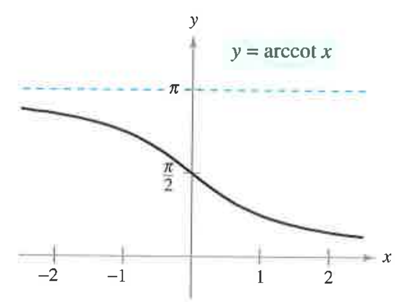
Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[-\pi/2, 0) \cup (0, \pi/2]$
Figure 5.29



Domain: $(-\infty, -1] \cup [1, \infty)$
Range: $[0, \pi/2) \cup (\pi/2, \pi]$



Domain: $(-\infty, \infty)$
Range: $(0, \pi)$

EXAMPLE 1 Evaluating Inverse Trigonometric Functions

Evaluate each function.

a. $\arcsin\left(-\frac{1}{2}\right)$ **b.** $\arccos 0$ **c.** $\arctan \sqrt{3}$ **d.** $\arcsin(0.3)$

NOTE When evaluating inverse trigonometric functions, remember that they denote *angles in radian measure.*

Solution

a. By definition, $y = \arcsin\left(-\frac{1}{2}\right)$ implies that $\sin y = -\frac{1}{2}$. In the interval $[-\pi/2, \pi/2]$, the correct value of y is $-\pi/6$.

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

b. By definition, $y = \arccos 0$ implies that $\cos y = 0$. In the interval $[0, \pi]$, you have $y = \pi/2$.

$$\arccos 0 = \frac{\pi}{2}$$

c. By definition, $y = \arctan \sqrt{3}$ implies that $\tan y = \sqrt{3}$. In the interval $(-\pi/2, \pi/2)$, you have $y = \pi/3$.

$$\arctan \sqrt{3} = \frac{\pi}{3}$$

d. Using a calculator set in *radian* mode produces

$$\arcsin(0.3) \approx 0.305.$$

EXPLORATION

Graph $y = \arccos(\cos x)$ for $-4\pi \le x \le 4\pi$. Why isn't the graph the same as the graph of $y = x$?

Inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

When applying these properties to inverse trigonometric functions, remember that the trigonometric functions have inverse functions only in restricted domains. For x-values outside these domains, these two properties do not hold. For example, $\arcsin(\sin \pi)$ is equal to 0, not π.

Properties of Inverse Trigonometric Functions

If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

If $|x| \ge 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$, then

$$\sec(\operatorname{arcsec} x) = x \quad \text{and} \quad \operatorname{arcsec}(\sec y) = y.$$

Similar properties hold for the other inverse trigonometric functions.

EXAMPLE 2 Solving an Equation

$$\arctan(2x - 3) = \frac{\pi}{4} \qquad \text{Original equation}$$

$$\tan[\arctan(2x - 3)] = \tan\frac{\pi}{4} \qquad \text{Take tangent of each side.}$$

$$2x - 3 = 1 \qquad \tan(\arctan x) = x$$

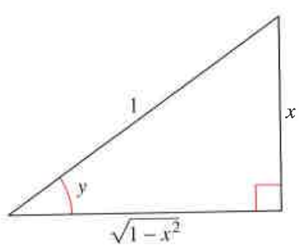
$$x = 2 \qquad \text{Solve for } x.$$

Some problems in calculus require that you evaluate expressions such as $\cos(\arcsin x)$, as shown in Example 3.

EXAMPLE 3 Using Right Triangles

a. Given $y = \arcsin x$, where $0 < y < \pi/2$, find $\cos y$.

b. Given $y = \operatorname{arcsec}(\sqrt{5}/2)$, find $\tan y$.



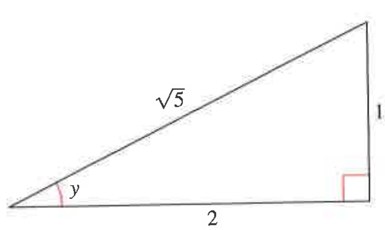
$y = \arcsin x$
Figure 5.30

Solution

a. Because $y = \arcsin x$, you know that $\sin y = x$. This relationship between x and y can be represented by a right triangle, as shown in Figure 5.30.

$$\cos y = \cos(\arcsin x) = \frac{\text{adj.}}{\text{hyp.}} = \sqrt{1 - x^2}$$

(This result is also valid for $-\pi/2 < y < 0$.)



$y = \operatorname{arcsec}\dfrac{\sqrt{5}}{2}$
Figure 5.31

b. Use the right triangle shown in Figure 5.31.

$$\tan y = \tan\left[\operatorname{arcsec}\left(\frac{\sqrt{5}}{2}\right)\right] = \frac{\text{opp.}}{\text{adj.}} = \frac{1}{2}$$

Derivatives of Inverse Trigonometric Functions

In Section 5.1 you saw that the derivative of the *transcendental* function $f(x) = \ln x$ is the *algebraic* function $f'(x) = 1/x$. You will now see that the derivatives of the inverse trigonometric functions also are algebraic (even though the inverse trigonometric functions are themselves transcendental).

The following theorem lists the derivatives of the six inverse trigonometric functions. Note that the derivatives of arccos u, arccot u, and arccsc u are the *negatives* of the derivatives of arcsin u, arctan u, and arcsec u, respectively.

NOTE There is no common agreement on the definition of arcsec x (or arccsc x) for negative values of x. When we defined the range of the arcsecant, we chose to preserve the reciprocal identity

$$\operatorname{arcsec} x = \arccos \frac{1}{x}.$$

For example, to evaluate arcsec(-2), you can write

$$\operatorname{arcsec}(-2) = \arccos(-0.5) \approx 2.09.$$

One of the consequences of the definition of the inverse secant function given in this text is that its graph has a positive slope at every x-value in its domain. (See Figure 5.29.) This accounts for the absolute value sign in the formula for the derivative of arcsec x.

THEOREM 5.16 Derivatives of Inverse Trigonometric Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

To derive these formulas, you can use implicit differentiation. For instance, if $y = \arcsin x$, then $\sin y = x$ and $(\cos y)y' = 1$. (See Exercise 94.)

TECHNOLOGY If your graphing utility does not have the arcsecant function, you can obtain its graph using

$$f(x) = \operatorname{arcsec} x = \arccos \frac{1}{x}.$$

EXAMPLE 4 Differentiating Inverse Trigonometric Functions

a. $\dfrac{d}{dx}[\arcsin(2x)] = \dfrac{2}{\sqrt{1-(2x)^2}} = \dfrac{2}{\sqrt{1-4x^2}}$

b. $\dfrac{d}{dx}[\arctan(3x)] = \dfrac{3}{1+(3x)^2} = \dfrac{3}{1+9x^2}$

c. $\dfrac{d}{dx}[\arcsin\sqrt{x}] = \dfrac{(1/2)\,x^{-1/2}}{\sqrt{1-x}} = \dfrac{1}{2\sqrt{x}\sqrt{1-x}} = \dfrac{1}{2\sqrt{x-x^2}}$

d. $\dfrac{d}{dx}[\operatorname{arcsec} e^{2x}] = \dfrac{2e^{2x}}{e^{2x}\sqrt{(e^{2x})^2-1}} = \dfrac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}} = \dfrac{2}{\sqrt{e^{4x}-1}}$

The absolute value sign is not necessary because $e^{2x} > 0$.

EXAMPLE 5 A Derivative That Can Be Simplified

Differentiate $y = \arcsin x + x\sqrt{1-x^2}$.

Solution

$$\begin{aligned} y' &= \frac{1}{\sqrt{1-x^2}} + x\left(\frac{1}{2}\right)(-2x)(1-x^2)^{-1/2} + \sqrt{1-x^2} \\ &= \frac{1}{\sqrt{1-x^2}} - \frac{x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} \\ &= \sqrt{1-x^2} + \sqrt{1-x^2} \\ &= 2\sqrt{1-x^2} \end{aligned}$$

NOTE From Example 5, you can see one of the benefits of inverse trigonometric functions—they can be used to integrate common algebraic functions. For instance, from the result shown in the example, it follows that

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right).$$

EXAMPLE 6 Analyzing an Inverse Trigonometric Graph

Analyze the graph of $y = (\arctan x)^2$.

Solution From the derivative

$$y' = 2(\arctan x)\left(\frac{1}{1+x^2}\right) = \frac{2 \arctan x}{1+x^2}$$

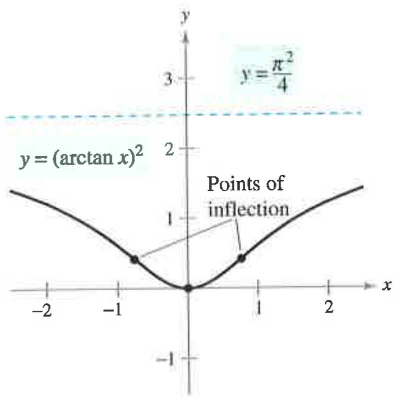
you can see that the only critical number is $x = 0$. By the First Derivative Test, this value corresponds to a relative minimum. From the second derivative

$$y'' = \frac{(1+x^2)\left(\frac{2}{1+x^2}\right) - (2\arctan x)(2x)}{(1+x^2)^2} = \frac{2(1 - 2x \arctan x)}{(1+x^2)^2}$$

it follows that points of inflection occur when $2x \arctan x = 1$. Using Newton's Method, these points occur when $x \approx \pm 0.765$. Finally, because

$$\lim_{x \to \pm\infty} (\arctan x)^2 = \frac{\pi^2}{4}$$

it follows that the graph has a horizontal asymptote at $y = \pi^2/4$. The graph is shown in Figure 5.32.



The graph of $y = (\arctan x)^2$ has a horizontal asymptote at $y = \pi^2/4$.
Figure 5.32

EXAMPLE 7 Maximizing an Angle

A photographer is taking a picture of a four-foot painting hung in an art gallery. The camera lens is 1 foot below the lower edge of the painting, as shown in Figure 5.33. How far should the camera be from the painting to maximize the angle subtended by the camera lens?

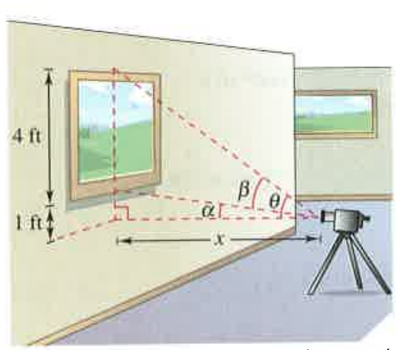
Solution In Figure 5.33, let β be the angle to be maximized.

$$\beta = \theta - \alpha = \operatorname{arccot}\frac{x}{5} - \operatorname{arccot} x$$

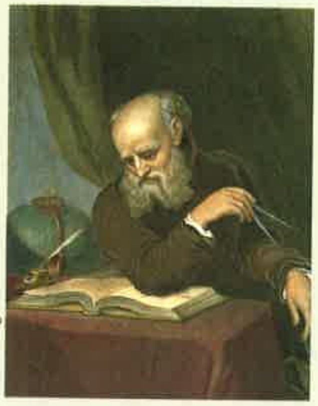
Differentiating produces

$$\frac{d\beta}{dx} = \frac{-1/5}{1 + (x^2/25)} - \frac{-1}{1+x^2} = \frac{-5}{25+x^2} + \frac{1}{1+x^2} = \frac{4(5-x^2)}{(25+x^2)(1+x^2)}.$$

Because $d\beta/dx = 0$ when $x = \sqrt{5}$, you can conclude from the First Derivative Test that this distance yields a maximum value of β. So, the distance is $x \approx 2.236$ feet and the angle is $\beta \approx 0.7297$ radian $\approx 41.81°$.



The camera should be 2.236 feet from the painting to maximize the angle β.
Figure 5.33



The Granger Collection

GALILEO GALILEI (1564–1642)

Galileo's approach to science departed from the accepted Aristotelian view that nature had describable *qualities*, such as "fluidity" and "potentiality." He chose to describe the physical world in terms of measurable *quantities*, such as time, distance, force, and mass.

Review of Basic Differentiation Rules

In the 1600s, Europe was ushered into the scientific age by such great thinkers as Descartes, Galileo, Huygens, Newton, and Kepler. These men believed that nature is governed by basic laws—laws that can, for the most part, be written in terms of mathematical equations. One of the most influential publications of this period—*Dialogue on the Great World Systems*, by Galileo Galilei—has become a classic description of modern scientific thought.

As mathematics has developed during the past few hundred years, a small number of elementary functions has proven sufficient for modeling most* phenomena in physics, chemistry, biology, engineering, economics, and a variety of other fields. An **elementary function** is a function from the following list or one that can be formed as the sum, product, quotient, or composition of functions in the list.

Algebraic Functions	*Transcendental Functions*
Polynomial functions	Logarithmic functions
Rational functions	Exponential functions
Functions involving radicals	Trigonometric functions
	Inverse trigonometric functions

With the differentiation rules introduced so far in the text, you can differentiate *any* elementary function. For convenience, these differentiation rules are summarized below.

Basic Differentiation Rules for Elementary Functions

1. $\frac{d}{dx}[cu] = cu'$
2. $\frac{d}{dx}[u \pm v] = u' \pm v'$
3. $\frac{d}{dx}[uv] = uv' + vu'$
4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}[c] = 0$
6. $\frac{d}{dx}[u^n] = nu^{n-1}u'$
7. $\frac{d}{dx}[x] = 1$
8. $\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), \quad u \neq 0$
9. $\frac{d}{dx}[\ln u] = \frac{u'}{u}$
10. $\frac{d}{dx}[e^u] = e^u u'$
11. $\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$
12. $\frac{d}{dx}[a^u] = (\ln a)a^u u'$
13. $\frac{d}{dx}[\sin u] = (\cos u)u'$
14. $\frac{d}{dx}[\cos u] = -(\sin u)u'$
15. $\frac{d}{dx}[\tan u] = (\sec^2 u)u'$
16. $\frac{d}{dx}[\cot u] = -(\csc^2 u)u'$
17. $\frac{d}{dx}[\sec u] = (\sec u \tan u)u'$
18. $\frac{d}{dx}[\csc u] = -(\csc u \cot u)u'$
19. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1 - u^2}}$
20. $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1 - u^2}}$
21. $\frac{d}{dx}[\arctan u] = \frac{u'}{1 + u^2}$
22. $\frac{d}{dx}[\text{arccot } u] = \frac{-u'}{1 + u^2}$
23. $\frac{d}{dx}[\text{arcsec } u] = \frac{u'}{|u|\sqrt{u^2 - 1}}$
24. $\frac{d}{dx}[\text{arccsc } u] = \frac{-u'}{|u|\sqrt{u^2 - 1}}$

* *Some important functions used in engineering and science (such as Bessel functions and gamma functions) are not elementary functions.*

Exercises for Section 5.6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

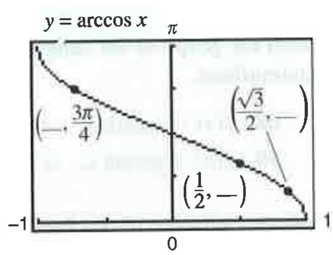
Numerical and Graphical Analysis **In Exercises 1 and 2, (a) use a graphing utility to complete the table, (b) plot the points in the table and graph the function by hand, (c) use a graphing utility to graph the function and compare the result with your hand-drawn graph in part (b), and (d) determine any intercepts and symmetry of the graph.**

x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y											

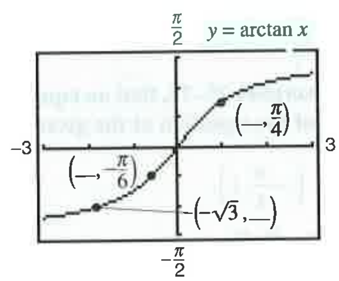
1. $y = \arcsin x$

2. $y = \arccos x$

3. Determine the missing coordinates of the points on the graph of the function.



4. Determine the missing coordinates of the points on the graph of the function.



In Exercises 5–12, evaluate the expression without using a calculator.

5. $\arcsin \frac{1}{2}$

6. $\arcsin 0$

7. $\arccos \frac{1}{2}$

8. $\arccos 0$

9. $\arctan \dfrac{\sqrt{3}}{3}$

10. $\operatorname{arccot}(-\sqrt{3})$

11. $\operatorname{arccsc}(-\sqrt{2})$

12. $\arccos\left(-\dfrac{\sqrt{3}}{2}\right)$

In Exercises 13–16, use a calculator to approximate the value. Round your answer to two decimal places.

13. $\arccos(-0.8)$

14. $\arcsin(-0.39)$

15. $\operatorname{arcsec} 1.269$

16. $\arctan(-3)$

In Exercises 17–20, evaluate each expression without using a calculator. (*Hint:* See Example 3.)

17. (a) $\sin\left(\arctan \dfrac{3}{4}\right)$

(b) $\sec\left(\arcsin \dfrac{4}{5}\right)$

18. (a) $\tan\left(\arccos \dfrac{\sqrt{2}}{2}\right)$

(b) $\cos\left(\arcsin \dfrac{5}{13}\right)$

19. (a) $\cot\left[\arcsin\left(-\dfrac{1}{2}\right)\right]$

(b) $\csc\left[\arctan\left(-\dfrac{5}{12}\right)\right]$

20. (a) $\sec\left[\arctan\left(-\dfrac{3}{5}\right)\right]$

(b) $\tan\left[\arcsin\left(-\dfrac{5}{6}\right)\right]$

In Exercises 21–28, write the expression in algebraic form.

21. $\cos(\arcsin 2x)$

22. $\sec(\arctan 4x)$

23. $\sin(\operatorname{arcsec} x)$

24. $\cos(\operatorname{arccot} x)$

25. $\tan\left(\operatorname{arcsec} \dfrac{x}{3}\right)$

26. $\sec[\arcsin(x - 1)]$

27. $\csc\left(\arctan \dfrac{x}{\sqrt{2}}\right)$

28. $\cos\left(\arcsin \dfrac{x - h}{r}\right)$

In Exercises 29 and 30, (a) use a graphing utility to graph f and g in the same viewing window to verify that they are equal, (b) use algebra to verify that f and g are equal, and (c) identify any horizontal asymptotes of the graphs.

29. $f(x) = \sin(\arctan 2x), \quad g(x) = \dfrac{2x}{\sqrt{1 + 4x^2}}$

30. $f(x) = \tan\left(\arccos \dfrac{x}{2}\right), \quad g(x) = \dfrac{\sqrt{4 - x^2}}{x}$

In Exercises 31–34, solve the equation for x.

31. $\arcsin(3x - \pi) = \frac{1}{2}$

32. $\arctan(2x - 5) = -1$

33. $\arcsin \sqrt{2x} = \arccos \sqrt{x}$

34. $\arccos x = \operatorname{arcsec} x$

In Exercises 35 and 36, verify each identity.

35. (a) $\operatorname{arccsc} x = \arcsin \dfrac{1}{x}, \quad x \geq 1$

(b) $\arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2}, \quad x > 0$

36. (a) $\arcsin(-x) = -\arcsin x, \quad |x| \leq 1$

(b) $\arccos(-x) = \pi - \arccos x, \quad |x| \leq 1$

In Exercises 37–40, sketch the graph of the function. Use a graphing utility to verify your graph.

37. $f(x) = \arcsin(x - 1)$

38. $f(x) = \arctan x + \dfrac{\pi}{2}$

39. $f(x) = \operatorname{arcsec} 2x$

40. $f(x) = \arccos \dfrac{x}{4}$

In Exercises 41–60, find the derivative of the function.

41. $f(x) = 2 \arcsin(x - 1)$

42. $f(t) = \arcsin t^2$

43. $g(x) = 3 \arccos \dfrac{x}{2}$

44. $f(x) = \operatorname{arcsec} 2x$

45. $f(x) = \arctan \dfrac{x}{a}$

46. $f(x) = \arctan \sqrt{x}$

47. $g(x) = \dfrac{\arcsin 3x}{x}$

48. $h(x) = x^2 \arctan x$

49. $h(t) = \sin(\arccos t)$

50. $f(x) = \arcsin x + \arccos x$

51. $y = x \arccos x - \sqrt{1 - x^2}$

52. $y = \ln(t^2 + 4) - \dfrac{1}{2} \arctan \dfrac{t}{2}$

53. $y = \dfrac{1}{2}\left(\dfrac{1}{2} \ln \dfrac{x + 1}{x - 1} + \arctan x\right)$

54. $y = \dfrac{1}{2}\left[x\sqrt{4 - x^2} + 4 \arcsin\left(\dfrac{x}{2}\right)\right]$

55. $y = x \arcsin x + \sqrt{1 - x^2}$

56. $y = x \arctan 2x - \dfrac{1}{4} \ln(1 + 4x^2)$

57. $y = 8 \arcsin \dfrac{x}{4} - \dfrac{x\sqrt{16 - x^2}}{2}$

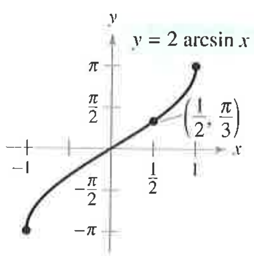
58. $y = 25 \arcsin \dfrac{x}{5} - x\sqrt{25 - x^2}$

59. $y = \arctan x + \dfrac{x}{1 + x^2}$

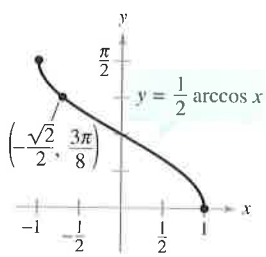
60. $y = \arctan \dfrac{x}{2} - \dfrac{1}{2(x^2 + 4)}$

In Exercises 61–66, find an equation of the tangent line to the graph of the function at the given point.

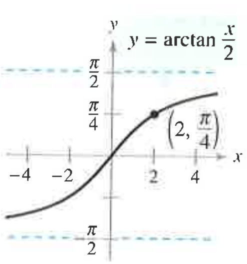
61. $y = 2 \arcsin x$



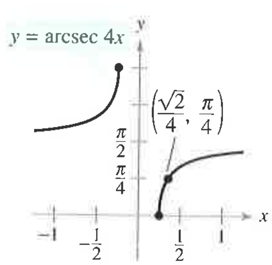
62. $y = \dfrac{1}{2} \arccos x$



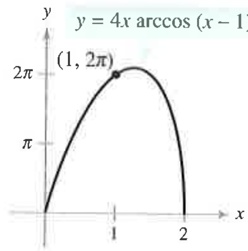
63. $y = \arctan \dfrac{x}{2}$



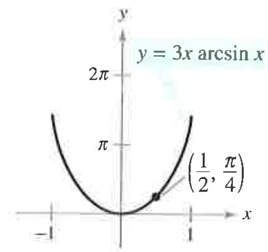
64. $y = \operatorname{arcsec} 4x$



65. $y = 4x \arccos(x - 1)$



66. $y = 3x \arcsin x$



Linear and Quadratic Approximations **In Exercises 67–70, use a computer algebra system to find the linear approximation**

$$P_1(x) = f(a) + f'(a)(x - a)$$

and the quadratic approximation

$$P_2(x) = f(a) + f'(a)(x - a) + \tfrac{1}{2}f''(a)(x - a)^2$$

of the function f at $x = a$. Sketch the graph of the function and its linear and quadratic approximations.

67. $f(x) = \arctan x, \quad a = 0$

68. $f(x) = \arccos x, \quad a = 0$

69. $f(x) = \arcsin x, \quad a = \frac{1}{2}$

70. $f(x) = \arctan x, \quad a = 1$

In Exercises 71–74, find any relative extrema of the function.

71. $f(x) = \operatorname{arcsec} x - x$

72. $f(x) = \arcsin x - 2x$

73. $f(x) = \arctan x - \arctan(x - 4)$

74. $h(x) = \arcsin x - 2 \arctan x$

Implicit Differentiation **In Exercises 75–78, find an equation of the tangent line to the graph of the equation at the given point.**

75. $x^2 + x \arctan y = y - 1, \quad \left(-\dfrac{\pi}{4}, 1\right)$

76. $\arctan(xy) = \arcsin(x + y), \quad (0, 0)$

77. $\arcsin x + \arcsin y = \dfrac{\pi}{2}, \quad \left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$

78. $\arctan(x + y) = y^2 + \dfrac{\pi}{4}, \quad (1, 0)$

Writing About Concepts

79. Explain why the domains of the trigonometric functions are restricted when finding the inverse trigonometric functions.

80. Explain why $\tan \pi = 0$ does not imply that $\arctan 0 = \pi$.

81. Explain how to graph $y = \operatorname{arccot} x$ on a graphing utility that does not have the arccotangent function.

82. Are the derivatives of the inverse trigonometric functions algebraic or transcendental functions? List the derivatives of the inverse trigonometric functions.

True or False? **In Exercises 83–88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

83. Because $\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$, it follows that $\arccos\frac{1}{2} = -\frac{\pi}{3}$.

84. $\arcsin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

85. The slope of the graph of the inverse tangent function is positive for all x.

86. The range of $y = \arcsin x$ is $[0, \pi]$.

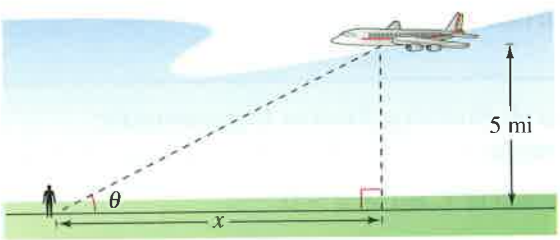
87. $\frac{d}{dx}[\arctan(\tan x)] = 1$ for all x in the domain.

88. $\arcsin^2 x + \arccos^2 x = 1$

89. ***Angular Rate of Change*** An airplane flies at an altitude of 5 miles toward a point directly over an observer. Consider θ and x as shown in the figure.

(a) Write θ as a function of x.

(b) The speed of the plane is 400 miles per hour. Find $d\theta/dt$ when $x = 10$ miles and $x = 3$ miles.



Not drawn to scale

90. ***Writing*** Repeat Exercise 89 if the altitude of the plane is 3 miles and describe how the altitude affects the rate of change of θ.

91. ***Angular Rate of Change*** In a free-fall experiment, an object is dropped from a height of 256 feet. A camera on the ground 500 feet from the point of impact records the fall of the object (see figure).

(a) Find the position function giving the height of the object at time t assuming the object is released at time $t = 0$. At what time will the object reach ground level?

(b) Find the rates of change of the angle of elevation of the camera when $t = 1$ and $t = 2$.

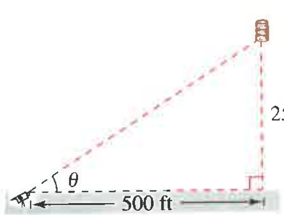


Figure for 91

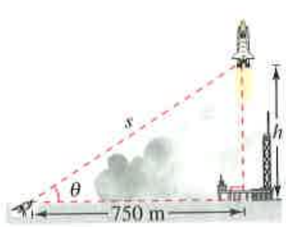


Figure for 92

92. ***Angular Rate of Change*** A television camera at ground level is filming the lift-off of a space shuttle at a point 750 meters from the launch pad. Let θ be the angle of elevation of the shuttle and let s be the distance between the camera and the shuttle (see figure). Write θ as a function of s for the period of time when the shuttle is moving vertically. Differentiate the result to find $d\theta/dt$ in terms of s and ds/dt.

93. (a) Prove that

$$\arctan x + \arctan y = \arctan\frac{x+y}{1-xy}, \quad xy \neq 1.$$

(b) Use the formula in part (a) to show that

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}.$$

94. Verify each differentiation formula.

(a) $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$ (b) $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$

(c) $\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}}$

(d) $\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$ (e) $\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$

(f) $\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$

95. ***Existence of an Inverse*** Determine the values of k such that the function $f(x) = kx + \sin x$ has an inverse function.

96. ***Think About It*** Use a graphing utility to graph $f(x) = \sin x$ and $g(x) = \arcsin(\sin x)$.

(a) Why isn't the graph of g the line $y = x$?

(b) Determine the extrema of g.

97. (a) Graph the function $f(x) = \arccos x + \arcsin x$ on the interval $[-1, 1]$. (b) Describe the graph of f. (c) Prove the result from part (b) analytically.

98. Prove that $\arcsin x = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)$, $|x| < 1$.

99. Find the value of c in the interval $[0, 4]$ on the x-axis that maximizes angle θ.

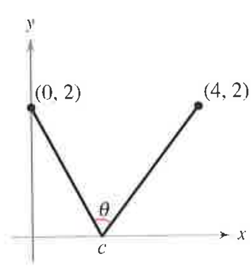


Figure for 99

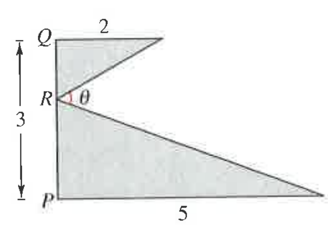


Figure for 100

100. Find PR such that $0 \leq PR \leq 3$ and $m\angle\theta$ is a maximum.

101. Some calculus textbooks define the inverse secant function using the range $[0, \pi/2) \cup [\pi, 3\pi/2)$.

(a) Sketch the graph $y = \operatorname{arcsec} x$ using this range.

(b) Show that $y' = \frac{1}{x\sqrt{x^2-1}}$.

Section 5.7 Inverse Trigonometric Functions: Integration

- Integrate functions whose antiderivatives involve inverse trigonometric functions.
- Use the method of completing the square to integrate a function.
- Review the basic integration rules involving elementary functions.

Integrals Involving Inverse Trigonometric Functions

The derivatives of the six inverse trigonometric functions fall into three pairs. In each pair, the derivative of one function is the negative of the other. For example,

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1 - x^2}}$$

and

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1 - x^2}}.$$

When listing the *antiderivative* that corresponds to each of the inverse trigonometric functions, you need to use only one member from each pair. It is conventional to use $\arcsin x$ as the antiderivative of $1/\sqrt{1 - x^2}$, rather than $-\arccos x$. The next theorem gives one antiderivative formula for each of the three pairs. The proofs of these integration rules are left to you (see Exercises 79–81).

FOR FURTHER INFORMATION For a detailed proof of part 2 of Theorem 5.17, see the article "A Direct Proof of the Integral Formula for Arctangent" by Arnold J. Insel in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

THEOREM 5.17 Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$
2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$
3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$

EXAMPLE 1 Integration with Inverse Trigonometric Functions

a. $$\int \frac{dx}{\sqrt{4 - x^2}} = \arcsin \frac{x}{2} + C$$

b. $$\int \frac{dx}{2 + 9x^2} = \frac{1}{3} \int \frac{3\,dx}{(\sqrt{2})^2 + (3x)^2} \qquad u = 3x,\ a = \sqrt{2}$$

$$= \frac{1}{3\sqrt{2}} \arctan \frac{3x}{\sqrt{2}} + C$$

c. $$\int \frac{dx}{x\sqrt{4x^2 - 9}} = \int \frac{2\,dx}{2x\sqrt{(2x)^2 - 3^2}} \qquad u = 2x,\ a = 3$$

$$= \frac{1}{3} \operatorname{arcsec} \frac{|2x|}{3} + C$$

The integrals in Example 1 are fairly straightforward applications of integration formulas. Unfortunately, this is not typical. The integration formulas for inverse trigonometric functions can be disguised in many ways.

TECHNOLOGY PITFALL

Computer software that can perform symbolic integration is useful for integrating functions such as the one in Example 2. When using such software, however, you must remember that it can fail to find an antiderivative for two reasons. First, some elementary functions simply do not have antiderivatives that are elementary functions. Second, every symbolic integration utility has limitations—you might have entered a function that the software was not programmed to handle. You should also remember that antiderivatives involving trigonometric functions or logarithmic functions can be written in many different forms. For instance, one symbolic integration utility found the integral in Example 2 to be

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \arctan \sqrt{e^{2x}-1} + C.$$

Try showing that this antiderivative is equivalent to that obtained in Example 2.

EXAMPLE 2 Integration by Substitution

Find $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$.

Solution As it stands, this integral doesn't fit any of the three inverse trigonometric formulas. Using the substitution $u = e^x$, however, produces

$$u = e^x \quad \Rightarrow \quad du = e^x\,dx \quad \Rightarrow \quad dx = \frac{du}{e^x} = \frac{du}{u}.$$

With this substitution, you can integrate as follows.

$$\begin{aligned}
\int \frac{dx}{\sqrt{e^{2x}-1}} &= \int \frac{dx}{\sqrt{(e^x)^2-1}} && \text{Write } e^{2x} \text{ as } (e^x)^2. \\
&= \int \frac{du/u}{\sqrt{u^2-1}} && \text{Substitute.} \\
&= \int \frac{du}{u\sqrt{u^2-1}} && \text{Rewrite to fit Arcsecant Rule.} \\
&= \operatorname{arcsec} \frac{|u|}{1} + C && \text{Apply Arcsecant Rule.} \\
&= \operatorname{arcsec} e^x + C && \text{Back-substitute.}
\end{aligned}$$

EXAMPLE 3 Rewriting as the Sum of Two Quotients

Find $\displaystyle\int \frac{x+2}{\sqrt{4-x^2}}\,dx$.

Solution This integral does not appear to fit any of the basic integration formulas. By splitting the integrand into two parts, however, you can see that the first part can be found with the Power Rule and the second part yields an inverse sine function.

$$\begin{aligned}
\int \frac{x+2}{\sqrt{4-x^2}}\,dx &= \int \frac{x}{\sqrt{4-x^2}}\,dx + \int \frac{2}{\sqrt{4-x^2}}\,dx \\
&= -\frac{1}{2}\int (4-x^2)^{-1/2}(-2x)\,dx + 2\int \frac{1}{\sqrt{4-x^2}}\,dx \\
&= -\frac{1}{2}\left[\frac{(4-x^2)^{1/2}}{1/2}\right] + 2 \arcsin \frac{x}{2} + C \\
&= -\sqrt{4-x^2} + 2 \arcsin \frac{x}{2} + C
\end{aligned}$$

Completing the Square

Completing the square helps when quadratic functions are involved in the integrand. For example, the quadratic $x^2 + bx + c$ can be written as the difference of two squares by adding and subtracting $(b/2)^2$.

$$\begin{aligned}
x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\
&= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c
\end{aligned}$$

EXAMPLE 4 Completing the Square

Find $\displaystyle\int \frac{dx}{x^2 - 4x + 7}$.

Solution You can write the denominator as the sum of two squares as shown.

$$\begin{aligned} x^2 - 4x + 7 &= (x^2 - 4x + 4) - 4 + 7 \\ &= (x - 2)^2 + 3 = u^2 + a^2 \end{aligned}$$

Now, in this completed square form, let $u = x - 2$ and $a = \sqrt{3}$.

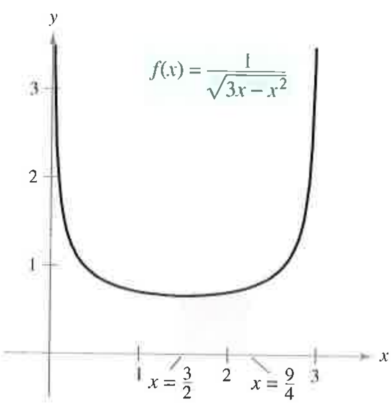
$$\int \frac{dx}{x^2 - 4x + 7} = \int \frac{dx}{(x-2)^2 + 3} = \frac{1}{\sqrt{3}} \arctan \frac{x-2}{\sqrt{3}} + C$$

If the leading coefficient is not 1, it helps to factor before completing the square. For instance, you can complete the square of $2x^2 - 8x + 10$ by factoring first.

$$\begin{aligned} 2x^2 - 8x + 10 &= 2(x^2 - 4x + 5) \\ &= 2(x^2 - 4x + 4 - 4 + 5) \\ &= 2[(x-2)^2 + 1] \end{aligned}$$

To complete the square when the coefficient of x^2 is negative, use the same factoring process shown above. For instance, you can complete the square for $3x - x^2$ as shown.

$$\begin{aligned} 3x - x^2 &= -(x^2 - 3x) \\ &= -\left[x^2 - 3x + \left(\tfrac{3}{2}\right)^2 - \left(\tfrac{3}{2}\right)^2\right] \\ &= \left(\tfrac{3}{2}\right)^2 - \left(x - \tfrac{3}{2}\right)^2 \end{aligned}$$



The area of the region bounded by the graph of f, the x-axis, $x = \frac{3}{2}$, and $x = \frac{9}{4}$ is $\pi/6$.
Figure 5.34

EXAMPLE 5 Completing the Square (Negative Leading Coefficient)

Find the area of the region bounded by the graph of

$$f(x) = \frac{1}{\sqrt{3x - x^2}}$$

the x-axis, and the lines $x = \frac{3}{2}$ and $x = \frac{9}{4}$.

Solution From Figure 5.34, you can see that the area is given by

$$\text{Area} = \int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}}\, dx.$$

Using the completed square form derived above, you can integrate as shown.

$$\begin{aligned} \int_{3/2}^{9/4} \frac{dx}{\sqrt{3x - x^2}} &= \int_{3/2}^{9/4} \frac{dx}{\sqrt{(3/2)^2 - [x - (3/2)]^2}} \\ &= \arcsin \frac{x - (3/2)}{3/2}\Bigg]_{3/2}^{9/4} \\ &= \arcsin \frac{1}{2} - \arcsin 0 \\ &= \frac{\pi}{6} \\ &\approx 0.524 \end{aligned}$$

TECHNOLOGY With definite integrals such as the one given in Example 5, remember that you can resort to a numerical solution. For instance, applying Simpson's Rule (with $n = 12$) to the integral in the example, you obtain

$$\int_{3/2}^{9/4} \frac{1}{\sqrt{3x - x^2}}\, dx \approx 0.523599.$$

This differs from the exact value of the integral ($\pi/6 \approx 0.5235988$) by less than one millionth.

Review of Basic Integration Rules

You have now completed the introduction of the **basic integration rules.** To be efficient at applying these rules, you should have practiced enough so that each rule is committed to memory.

Basic Integration Rules ($a > 0$)

1. $\int kf(u)\,du = k\int f(u)\,du$
2. $\int [f(u) \pm g(u)]\,du = \int f(u)\,du \pm \int g(u)\,du$
3. $\int du = u + C$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C, \quad n \neq -1$
5. $\int \dfrac{du}{u} = \ln|u| + C$
6. $\int e^u\,du = e^u + C$
7. $\int a^u\,du = \left(\dfrac{1}{\ln a}\right)a^u + C$
8. $\int \sin u\,du = -\cos u + C$
9. $\int \cos u\,du = \sin u + C$
10. $\int \tan u\,du = -\ln|\cos u| + C$
11. $\int \cot u\,du = \ln|\sin u| + C$
12. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
13. $\int \csc u\,du = -\ln|\csc u + \cot u| + C$
14. $\int \sec^2 u\,du = \tan u + C$
15. $\int \csc^2 u\,du = -\cot u + C$
16. $\int \sec u \tan u\,du = \sec u + C$
17. $\int \csc u \cot u\,du = -\csc u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \arcsin \dfrac{u}{a} + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\arctan \dfrac{u}{a} + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\operatorname{arcsec} \dfrac{|u|}{a} + C$

You can learn a lot about the nature of integration by comparing this list with the summary of differentiation rules given in the preceding section. For differentiation, you now have rules that allow you to differentiate *any* elementary function. For integration, this is far from true.

The integration rules listed above are primarily those that were happened on when developing differentiation rules. So far, you have not learned any rules or techniques for finding the antiderivative of a general product or quotient, the natural logarithmic function, or the inverse trigonometric functions. More importantly, you cannot apply any of the rules in this list unless you can create the proper du corresponding to the u in the formula. The point is that you need to work more on integration techniques, which you will do in Chapter 8. The next two examples should give you a better feeling for the integration problems that you *can* and *cannot* do with the techniques and rules you now know.

EXAMPLE 6 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$ **b.** $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-1}}$ **c.** $\displaystyle\int \frac{dx}{\sqrt{x^2-1}}$

Solution

a. You *can* find this integral (it fits the Arcsecant Rule).

$$\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec}|x| + C$$

b. You *can* find this integral (it fits the Power Rule).

$$\begin{aligned}\int \frac{x\,dx}{\sqrt{x^2-1}} &= \frac{1}{2}\int (x^2-1)^{-1/2}(2x)\,dx \\ &= \frac{1}{2}\left[\frac{(x^2-1)^{1/2}}{1/2}\right] + C \\ &= \sqrt{x^2-1} + C\end{aligned}$$

c. You *cannot* find this integral using present techniques. (You should scan the list of basic integration rules to verify this conclusion.)

EXAMPLE 7 Comparing Integration Problems

Find as many of the following integrals as you can using the formulas and techniques you have studied so far in the text.

a. $\displaystyle\int \frac{dx}{x\ln x}$ **b.** $\displaystyle\int \frac{\ln x\,dx}{x}$ **c.** $\displaystyle\int \ln x\,dx$

Solution

a. You *can* find this integral (it fits the Log Rule).

$$\begin{aligned}\int \frac{dx}{x\ln x} &= \int \frac{1/x}{\ln x}\,dx \\ &= \ln|\ln x| + C\end{aligned}$$

b. You *can* find this integral (it fits the Power Rule).

$$\begin{aligned}\int \frac{\ln x\,dx}{x} &= \int \left(\frac{1}{x}\right)(\ln x)^1\,dx \\ &= \frac{(\ln x)^2}{2} + C\end{aligned}$$

c. You *cannot* find this integral using present techniques.

NOTE Note in Examples 6 and 7 that the *simplest* functions are the ones that you cannot yet integrate.

Exercises for Section 5.7

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–20, find the integral.

1. $\int \frac{5}{\sqrt{9-x^2}}\,dx$
2. $\int \frac{3}{\sqrt{1-4x^2}}\,dx$
3. $\int \frac{7}{16+x^2}\,dx$
4. $\int \frac{4}{1+9x^2}\,dx$
5. $\int \frac{1}{x\sqrt{4x^2-1}}\,dx$
6. $\int \frac{1}{4+(x-1)^2}\,dx$
7. $\int \frac{x^3}{x^2+1}\,dx$
8. $\int \frac{x^4-1}{x^2+1}\,dx$
9. $\int \frac{1}{\sqrt{1-(x+1)^2}}\,dx$
10. $\int \frac{t}{t^4+16}\,dt$
11. $\int \frac{t}{\sqrt{1-t^4}}\,dt$
12. $\int \frac{1}{x\sqrt{x^4-4}}\,dx$
13. $\int \frac{e^{2x}}{4+e^{4x}}\,dx$
14. $\int \frac{1}{3+(x-2)^2}\,dx$
15. $\int \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$
16. $\int \frac{3}{2\sqrt{x}(1+x)}\,dx$
17. $\int \frac{x-3}{x^2+1}\,dx$
18. $\int \frac{4x+3}{\sqrt{1-x^2}}\,dx$
19. $\int \frac{x+5}{\sqrt{9-(x-3)^2}}\,dx$
20. $\int \frac{x-2}{(x+1)^2+4}\,dx$

In Exercises 21–30, evaluate the integral.

21. $\int_0^{1/6} \frac{1}{\sqrt{1-9x^2}}\,dx$
22. $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$
23. $\int_0^{\sqrt{3}/2} \frac{1}{1+4x^2}\,dx$
24. $\int_{\sqrt{3}}^3 \frac{1}{9+x^2}\,dx$
25. $\int_0^{1/\sqrt{2}} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$
26. $\int_0^{1/\sqrt{2}} \frac{\arccos x}{\sqrt{1-x^2}}\,dx$
27. $\int_{-1/2}^0 \frac{x}{\sqrt{1-x^2}}\,dx$
28. $\int_{-\sqrt{3}}^0 \frac{x}{1+x^2}\,dx$
29. $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$
30. $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx$

In Exercises 31–42, find or evaluate the integral. (Complete the square, if necessary.)

31. $\int_0^2 \frac{dx}{x^2-2x+2}$
32. $\int_{-2}^2 \frac{dx}{x^2+4x+13}$
33. $\int \frac{2x}{x^2+6x+13}\,dx$
34. $\int \frac{2x-5}{x^2+2x+2}\,dx$
35. $\int \frac{1}{\sqrt{-x^2-4x}}\,dx$
36. $\int \frac{2}{\sqrt{-x^2+4x}}\,dx$
37. $\int \frac{x+2}{\sqrt{-x^2-4x}}\,dx$
38. $\int \frac{x-1}{\sqrt{x^2-2x}}\,dx$
39. $\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}\,dx$
40. $\int \frac{1}{(x-1)\sqrt{x^2-2x}}\,dx$
41. $\int \frac{x}{x^4+2x^2+2}\,dx$
42. $\int \frac{x}{\sqrt{9+8x^2-x^4}}\,dx$

In Exercises 43–46, use the specified substitution to find or evaluate the integral.

43. $\int \sqrt{e^t-3}\,dt$
$u = \sqrt{e^t-3}$
44. $\int \frac{\sqrt{x-2}}{x+1}\,dx$
$u = \sqrt{x-2}$
45. $\int_1^3 \frac{dx}{\sqrt{x}(1+x)}$
$u = \sqrt{x}$
46. $\int_0^1 \frac{dx}{2\sqrt{3-x}\sqrt{x+1}}$
$u = \sqrt{x+1}$

Writing About Concepts

In Exercises 47–50, determine which of the integrals can be found using the basic integration formulas you have studied so far in the text.

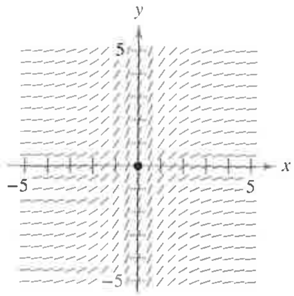
47. (a) $\int \frac{1}{\sqrt{1-x^2}}\,dx$ (b) $\int \frac{x}{\sqrt{1-x^2}}\,dx$ (c) $\int \frac{1}{x\sqrt{1-x^2}}\,dx$
48. (a) $\int e^{x^2}\,dx$ (b) $\int xe^{x^2}\,dx$ (c) $\int \frac{1}{x^2}e^{1/x}\,dx$
49. (a) $\int \sqrt{x-1}\,dx$ (b) $\int x\sqrt{x-1}\,dx$ (c) $\int \frac{x}{\sqrt{x-1}}\,dx$
50. (a) $\int \frac{1}{1+x^4}\,dx$ (b) $\int \frac{x}{1+x^4}\,dx$ (c) $\int \frac{x^3}{1+x^4}\,dx$
51. Determine which value best approximates the area of the region between the x-axis and the function
$$f(x) = \frac{1}{\sqrt{1-x^2}}$$
over the interval $[-0.5, 0.5]$. (Make your selection on the basis of a sketch of the region and *not* by performing any calculations.)
(a) 4 (b) -3 (c) 1 (d) 2 (e) 3
52. Decide whether you can find the integral
$$\int \frac{2\,dx}{\sqrt{x^2+4}}$$
using the formulas and techniques you have studied so far. Explain your reasoning.

Differential Equations **In Exercises 53 and 54, use the differential equation and the specified initial condition to find y.**

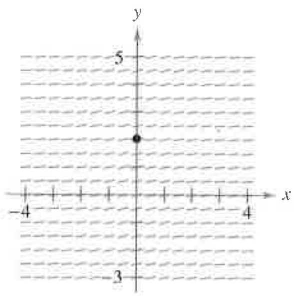
53. $\frac{dy}{dx} = \frac{1}{\sqrt{4-x^2}}$
$y(0) = \pi$
54. $\frac{dy}{dx} = \frac{1}{4+x^2}$
$y(2) = \pi$

Slope Fields **In Exercises 55–58, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a). To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

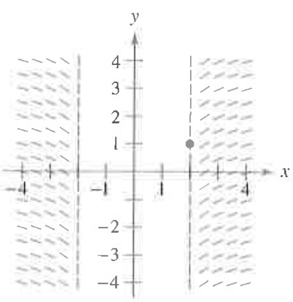
55. $\dfrac{dy}{dx} = \dfrac{3}{1 + x^2}$, $(0, 0)$



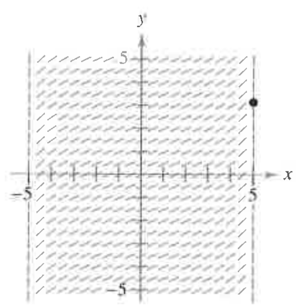
56. $\dfrac{dy}{dx} = \dfrac{2}{9 + x^2}$, $(0, 2)$



57. $\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2 - 4}}$, $(2, 1)$



58. $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{25 - x^2}}$, $(5, \pi)$



Slope Fields **In Exercises 59–62, use a computer algebra system to graph the slope field for the differential equation and graph the solution satisfying the specified initial condition.**

59. $\dfrac{dy}{dx} = \dfrac{10}{x\sqrt{x^2 - 1}}$

$y(3) = 0$

60. $\dfrac{dy}{dx} = \dfrac{1}{12 + x^2}$

$y(4) = 2$

61. $\dfrac{dy}{dx} = \dfrac{2y}{\sqrt{16 - x^2}}$

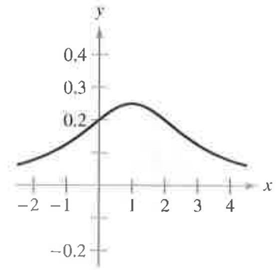
$y(0) = 2$

62. $\dfrac{dy}{dx} = \dfrac{\sqrt{y}}{1 + x^2}$

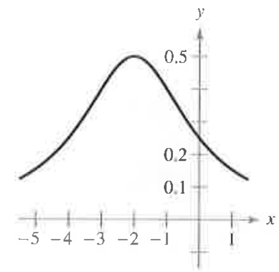
$y(0) = 4$

Area **In Exercises 63–68, find the area of the region.**

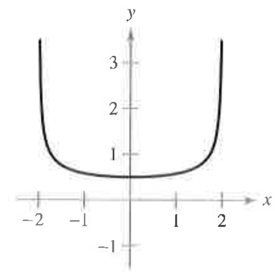
63. $y = \dfrac{1}{x^2 - 2x + 5}$



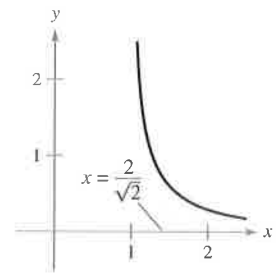
64. $y = \dfrac{2}{x^2 + 4x + 8}$



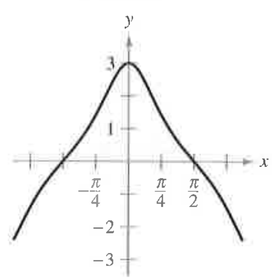
65. $y = \dfrac{1}{\sqrt{4 - x^2}}$



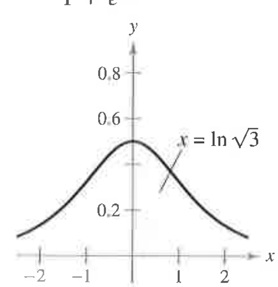
66. $y = \dfrac{1}{x\sqrt{x^2 - 1}}$



67. $y = \dfrac{3\cos x}{1 + \sin^2 x}$



68. $y = \dfrac{e^x}{1 + e^{2x}}$



In Exercises 69 and 70, (a) verify the integration formula, then (b) use it to find the area of the region.

69. $\displaystyle\int \frac{\arctan x}{x^2}\,dx = \ln x - \frac{1}{2}\ln(1 + x^2) - \frac{\arctan x}{x} + C$

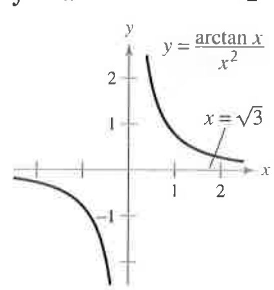


Figure for 69

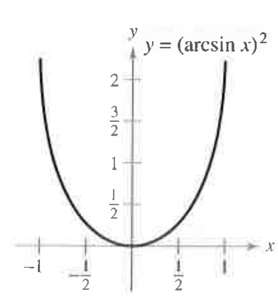


Figure for 70

70. $\displaystyle\int (\arcsin x)^2\,dx$

$= x(\arcsin x)^2 - 2x + 2\sqrt{1 - x^2}\arcsin x + C$

71. (a) Sketch the region whose area is represented by

$$\int_0^1 \arcsin x \, dx.$$

(b) Use the integration capabilities of a graphing utility to approximate the area.

(c) Find the exact area analytically.

72. (a) Show that $\displaystyle\int_0^1 \frac{4}{1+x^2}\, dx = \pi.$

(b) Approximate the number π using Simpson's Rule (with $n = 6$) and the integral in part (a).

(c) Approximate the number π by using the integration capabilities of a graphing utility.

73. ***Investigation*** Consider the function $F(x) = \displaystyle\frac{1}{2}\int_x^{x+2} \frac{2}{t^2+1}\, dt.$

(a) Write a short paragraph giving a geometric interpretation of the function $F(x)$ relative to the function $f(x) = \dfrac{2}{x^2+1}$. Use what you have written to guess the value of x that will make F maximum.

(b) Perform the specified integration to find an alternative form of $F(x)$. Use calculus to locate the value of x that will make F maximum and compare the result with your guess in part (a).

74. Consider the integral $\displaystyle\int \frac{1}{\sqrt{6x - x^2}}\, dx.$

(a) Find the integral by completing the square of the radicand.

(b) Find the integral by making the substitution $u = \sqrt{x}$.

(c) The antiderivatives in parts (a) and (b) appear to be significantly different. Use a graphing utility to graph each antiderivative in the same viewing window and determine the relationship between them. Find the domain of each.

True or False? **In Exercises 75–78, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

75. $\displaystyle\int \frac{dx}{3x\sqrt{9x^2-16}} = \frac{1}{4}\operatorname{arcsec}\frac{3x}{4} + C$

76. $\displaystyle\int \frac{dx}{25+x^2} = \frac{1}{25}\arctan\frac{x}{25} + C$

77. $\displaystyle\int \frac{dx}{\sqrt{4-x^2}} = -\arccos\frac{x}{2} + C$

78. One way to find $\displaystyle\int \frac{2e^{2x}}{\sqrt{9-e^{2x}}}\, dx$ is to use the Arcsine Rule.

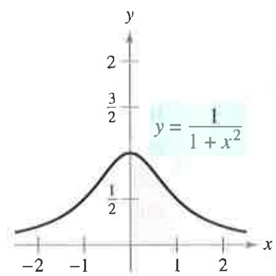
Verifying Integration Rules **In Exercises 79–81, verify each rule by differentiating. Let $a > 0$.**

79. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$

80. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$

81. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

82. ***Numerical Integration*** (a) Write an integral that represents the area of the region. (b) Then use the Trapezoidal Rule with $n = 8$ to estimate the area of the region. (c) Explain how you can use the results of parts (a) and (b) to estimate π.



83. ***Vertical Motion*** An object is projected upward from ground level with an initial velocity of 500 feet per second. In this exercise, the goal is to analyze the motion of the object during its upward flight.

(a) If air resistance is neglected, find the velocity of the object as a function of time. Use a graphing utility to graph this function.

(b) Use the result in part (a) to find the position function and determine the maximum height attained by the object.

(c) If the air resistance is proportional to the square of the velocity, you obtain the equation

$$\frac{dv}{dt} = -(32 + kv^2)$$

where -32 feet per second per second is the acceleration due to gravity and k is a constant. Find the velocity as a function of time by solving the equation

$$\int \frac{dv}{32+kv^2} = -\int dt.$$

(d) Use a graphing utility to graph the velocity function $v(t)$ in part (c) if $k = 0.001$. Use the graph to approximate the time t_0 at which the object reaches its maximum height.

(e) Use the integration capabilities of a graphing utility to approximate the integral

$$\int_0^{t_0} v(t)\, dt$$

where $v(t)$ and t_0 are those found in part (d). This is the approximation of the maximum height of the object.

(f) Explain the difference between the results in parts (b) and (e).

FOR FURTHER INFORMATION For more information on this topic, see "What Goes Up Must Come Down; Will Air Resistance Make It Return Sooner, or Later?" by John Lekner in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

84. Graph $y_1 = \dfrac{x}{1+x^2}$, $y_2 = \arctan x$, and $y_3 = x$ on $[0, 10]$.

Prove that $\dfrac{x}{1+x^2} < \arctan x < x$ for $x > 0$.

Section 5.8 Hyperbolic Functions

- **Develop properties of hyperbolic functions.**
- **Differentiate and integrate hyperbolic functions.**
- **Develop properties of inverse hyperbolic functions.**
- **Differentiate and integrate functions involving inverse hyperbolic functions.**

JOHANN HEINRICH LAMBERT (1728–1777)

The first person to publish a comprehensive study on hyperbolic functions was Johann Heinrich Lambert, a Swiss-German mathematician and colleague of Euler.

Hyperbolic Functions

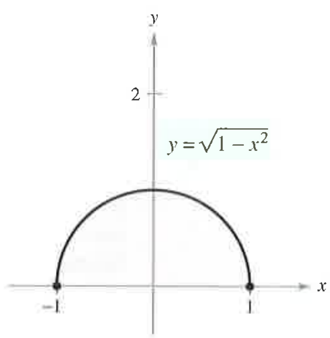
In this section you will look briefly at a special class of exponential functions called **hyperbolic functions.** The name *hyperbolic function* arose from comparison of the area of a semicircular region, as shown in Figure 5.35, with the area of a region under a hyperbola, as shown in Figure 5.36. The integral for the semicircular region involves an inverse trigonometric (circular) function:

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx = \frac{1}{2}\left[x\sqrt{1-x^2} + \arcsin x\right]_{-1}^{1} = \frac{\pi}{2} \approx 1.571.$$

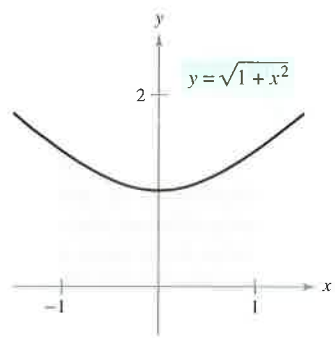
The integral for the hyperbolic region involves an inverse hyperbolic function:

$$\int_{-1}^{1} \sqrt{1+x^2}\,dx = \frac{1}{2}\left[x\sqrt{1+x^2} + \sinh^{-1}x\right]_{-1}^{1} \approx 2.296.$$

This is only one of many ways in which the hyperbolic functions are similar to the trigonometric functions.



Circle: $x^2 + y^2 = 1$
Figure 5.35



Hyperbola: $-x^2 + y^2 = 1$
Figure 5.36

FOR FURTHER INFORMATION For more information on the development of hyperbolic functions, see the article "An Introduction to Hyperbolic Functions in Elementary Calculus" by Jerome Rosenthal in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com.*

Definitions of the Hyperbolic Functions

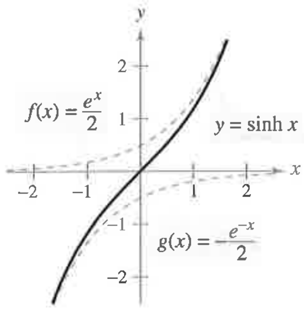
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}, \quad x \neq 0$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

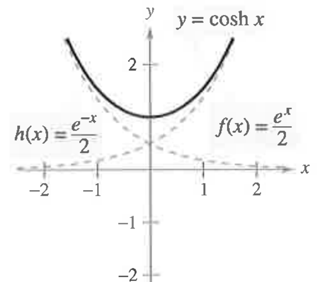
$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{1}{\tanh x}, \quad x \neq 0$$

NOTE $\sinh x$ is read as "the hyperbolic sine of x," $\cosh x$ as "the hyperbolic cosine of x," and so on.

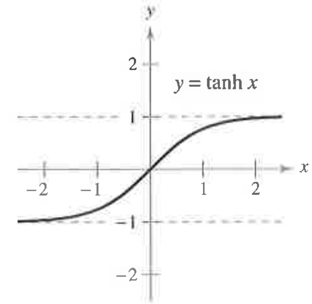
The graphs of the six hyperbolic functions and their domains and ranges are shown in Figure 5.37. Note that the graph of $\sinh x$ can be obtained by *addition of ordinates* using the exponential functions $f(x) = \frac{1}{2}e^x$ and $g(x) = -\frac{1}{2}e^{-x}$. Likewise, the graph of $\cosh x$ can be obtained by *addition of ordinates* using the exponential functions $f(x) = \frac{1}{2}e^x$ and $h(x) = \frac{1}{2}e^{-x}$.



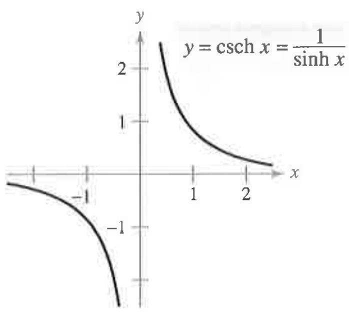
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



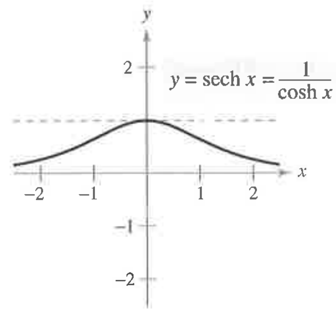
Domain: $(-\infty, \infty)$
Range: $[1, \infty)$



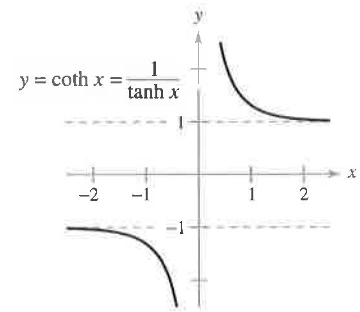
Domain: $(-\infty, \infty)$
Range: $(-1, 1)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$



Domain: $(-\infty, \infty)$
Range: $(0, 1]$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, -1) \cup (1, \infty)$

Figure 5.37

Many of the trigonometric identities have corresponding *hyperbolic identities*. For instance,

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\
&= \frac{4}{4} \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
2 \sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{e^{2x} - e^{-2x}}{2} \\
&= \sinh 2x.
\end{aligned}$$

FOR FURTHER INFORMATION To understand geometrically the relationship between the hyperbolic and exponential functions, see the article "A Short Proof Linking the Hyperbolic and Exponential Functions" by Michael J. Seery in *The AMATYC Review.*

Hyperbolic Identities

$\cosh^2 x - \sinh^2 x = 1$

$\tanh^2 x + \operatorname{sech}^2 x = 1$

$\coth^2 x - \operatorname{csch}^2 x = 1$

$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$

$\sinh^2 x = \dfrac{-1 + \cosh 2x}{2}$

$\cosh^2 x = \dfrac{1 + \cosh 2x}{2}$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

Differentiation and Integration of Hyperbolic Functions

Because the hyperbolic functions are written in terms of e^x and e^{-x}, you can easily derive rules for their derivatives. The following theorem lists these derivatives with the corresponding integration rules.

THEOREM 5.18 Derivatives and Integrals of Hyperbolic Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\sinh u] = (\cosh u)u' \qquad \int \cosh u\, du = \sinh u + C$$

$$\frac{d}{dx}[\cosh u] = (\sinh u)u' \qquad \int \sinh u\, du = \cosh u + C$$

$$\frac{d}{dx}[\tanh u] = (\operatorname{sech}^2 u)u' \qquad \int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\frac{d}{dx}[\coth u] = -(\operatorname{csch}^2 u)u' \qquad \int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\frac{d}{dx}[\operatorname{sech} u] = -(\operatorname{sech} u \tanh u)u' \qquad \int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\frac{d}{dx}[\operatorname{csch} u] = -(\operatorname{csch} u \coth u)u' \qquad \int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

Proof

$$\begin{aligned}\frac{d}{dx}[\sinh x] &= \frac{d}{dx}\left[\frac{e^x - e^{-x}}{2}\right] \\ &= \frac{e^x + e^{-x}}{2} = \cosh x \\ \frac{d}{dx}[\tanh x] &= \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right] \\ &= \frac{\cosh x(\cosh x) - \sinh x(\sinh x)}{\cosh^2 x} \\ &= \frac{1}{\cosh^2 x} \\ &= \operatorname{sech}^2 x\end{aligned}$$

In Exercises 98 and 102, you are asked to prove some of the other differentiation rules.

EXAMPLE 1 Differentiation of Hyperbolic Functions

a. $\dfrac{d}{dx}[\sinh(x^2 - 3)] = 2x\cosh(x^2 - 3)$ **b.** $\dfrac{d}{dx}[\ln(\cosh x)] = \dfrac{\sinh x}{\cosh x} = \tanh x$

c. $\dfrac{d}{dx}[x\sinh x - \cosh x] = x\cosh x + \sinh x - \sinh x = x\cosh x$

EXAMPLE 2 Finding Relative Extrema

Find the relative extrema of $f(x) = (x - 1)\cosh x - \sinh x$.

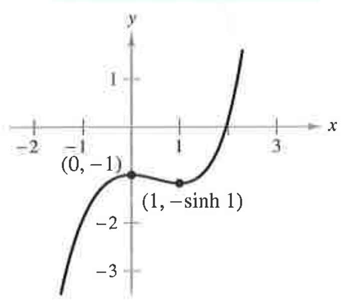
Solution Begin by setting the first derivative of f equal to 0.

$$f'(x) = (x - 1)\sinh x + \cosh x - \cosh x = 0$$
$$(x - 1)\sinh x = 0$$

So, the critical numbers are $x = 1$ and $x = 0$. Using the Second Derivative Test, you can verify that the point $(0, -1)$ yields a relative maximum and the point $(1, -\sinh 1)$ yields a relative minimum, as shown in Figure 5.38. Try using a graphing utility to confirm this result. If your graphing utility does not have hyperbolic functions, you can use exponential functions as follows.

$$\begin{aligned} f(x) &= (x - 1)\left(\tfrac{1}{2}\right)(e^x + e^{-x}) - \tfrac{1}{2}(e^x - e^{-x}) \\ &= \tfrac{1}{2}(xe^x + xe^{-x} - e^x - e^{-x} - e^x + e^{-x}) \\ &= \tfrac{1}{2}(xe^x + xe^{-x} - 2e^x) \end{aligned}$$

$f(x) = (x - 1)\cosh x - \sinh x$



$f''(0) < 0$, so $(0, -1)$ is a relative maximum. $f''(1) > 0$, so $(1, -\sinh 1)$ is a relative minimum.
Figure 5.38

When a uniform flexible cable, such as a telephone wire, is suspended from two points, it takes the shape of a *catenary*, as discussed in Example 3.

EXAMPLE 3 Hanging Power Cables

Power cables are suspended between two towers, forming the catenary shown in Figure 5.39. The equation for this catenary is

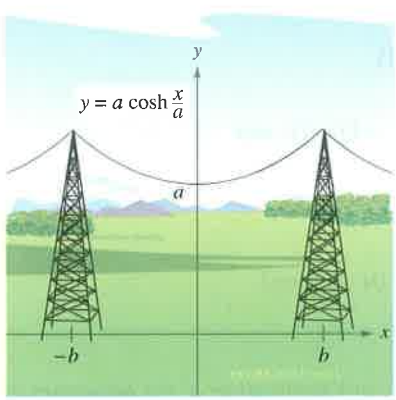
$$y = a\cosh\frac{x}{a}.$$

The distance between the two towers is $2b$. Find the slope of the catenary at the point where the cable meets the right-hand tower.

Solution Differentiating produces

$$y' = a\left(\frac{1}{a}\right)\sinh\frac{x}{a} = \sinh\frac{x}{a}.$$

At the point $(b, a\cosh(b/a))$, the slope (from the left) is given by $m = \sinh\dfrac{b}{a}$.



Catenary
Figure 5.39

FOR FURTHER INFORMATION In Example 3, the cable is a catenary between two supports at the same height. To learn about the shape of a cable hanging between supports of different heights, see the article "Reexamining the Catenary" by Paul Cella in *The College Mathematics Journal*. To view this article, go to the website *www.matharticles.com*.

EXAMPLE 4 Integrating a Hyperbolic Function

Find $\int \cosh 2x \sinh^2 2x\, dx$.

Solution

$$\begin{aligned}\int \cosh 2x \sinh^2 2x\, dx &= \frac{1}{2}\int (\sinh 2x)^2(2\cosh 2x)\, dx && u = \sinh 2x \\ &= \frac{1}{2}\left[\frac{(\sinh 2x)^3}{3}\right] + C \\ &= \frac{\sinh^3 2x}{6} + C\end{aligned}$$

Inverse Hyperbolic Functions

Unlike trigonometric functions, hyperbolic functions are not periodic. In fact, by looking back at Figure 5.37, you can see that four of the six hyperbolic functions are actually one-to-one (the hyperbolic sine, tangent, cosecant, and cotangent). So, you can apply Theorem 5.7 to conclude that these four functions have inverse functions. The other two (the hyperbolic cosine and secant) are one-to-one if their domains are restricted to the positive real numbers, and for this restricted domain they also have inverse functions. Because the hyperbolic functions are defined in terms of exponential functions, it is not surprising to find that the inverse hyperbolic functions can be written in terms of logarithmic functions, as shown in Theorem 5.19.

THEOREM 5.19 Inverse Hyperbolic Functions

Function	*Domain*
$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$	$(-\infty, \infty)$
$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$	$[1, \infty)$
$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$	$(-1, 1)$
$\coth^{-1} x = \frac{1}{2}\ln\frac{x+1}{x-1}$	$(-\infty, -1) \cup (1, \infty)$
$\operatorname{sech}^{-1} x = \ln\frac{1+\sqrt{1-x^2}}{x}$	$(0, 1]$
$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{\lvert x\rvert}\right)$	$(-\infty, 0) \cup (0, \infty)$

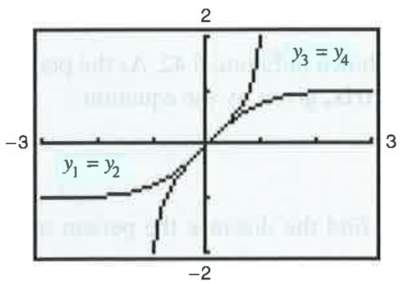
Proof The proof of this theorem is a straightforward application of the properties of the exponential and logarithmic functions. For example, if

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

and

$$g(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

you can show that $f(g(x)) = x$ and $g(f(x)) = x$, which implies that g is the inverse function of f.



Graphs of the hyperbolic tangent function and the inverse hyperbolic tangent function
Figure 5.40

TECHNOLOGY You can use a graphing utility to confirm graphically the results of Theorem 5.19. For instance, graph the following functions.

$$y_1 = \tanh x \qquad \text{Hyperbolic tangent}$$

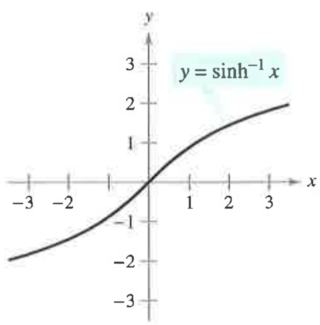
$$y_2 = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \text{Definition of hyperbolic tangent}$$

$$y_3 = \tanh^{-1} x \qquad \text{Inverse hyperbolic tangent}$$

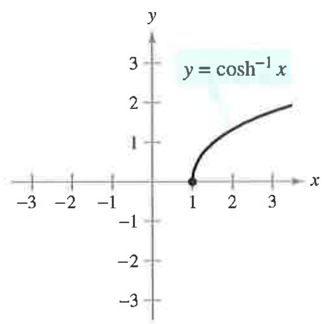
$$y_4 = \frac{1}{2}\ln\frac{1 + x}{1 - x} \qquad \text{Definition of inverse hyperbolic tangent}$$

The resulting display is shown in Figure 5.40. As you watch the graphs being traced out, notice that $y_1 = y_2$ and $y_3 = y_4$. Also notice that the graph of y_1 is the reflection of the graph of y_3 in the line $y = x$.

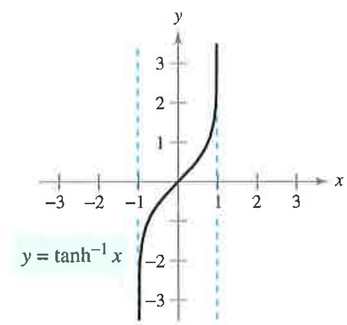
The graphs of the inverse hyperbolic functions are shown in Figure 5.41.



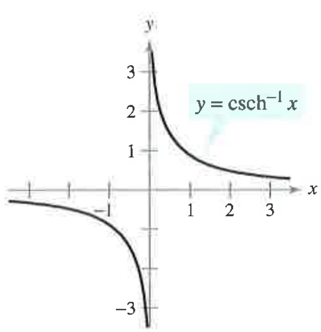
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



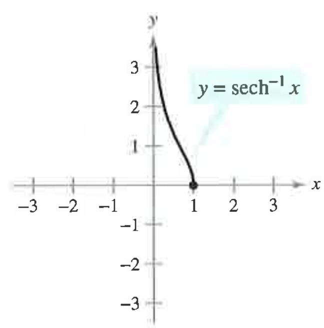
Domain: $[1, \infty)$
Range: $[0, \infty)$



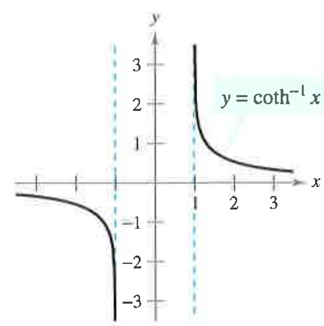
Domain: $(-1, 1)$
Range: $(-\infty, \infty)$



Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$
Figure 5.41



Domain: $(0, 1]$
Range: $[0, \infty)$



Domain: $(-\infty, -1) \cup (1, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$

The inverse hyperbolic secant can be used to define a curve called a *tractrix* or *pursuit curve*, as discussed in Example 5.

92. ***Vertical Motion*** An object is dropped from a height of 400 feet.

(a) Find the velocity of the object as a function of time (neglect air resistance on the object).

(b) Use the result in part (a) to find the position function.

(c) If the air resistance is proportional to the square of the velocity, then $dv/dt = -32 + kv^2$, where -32 feet per second per second is the acceleration due to gravity and k is a constant. Show that the velocity v as a function of time is

$$v(t) = -\sqrt{\frac{32}{k}}\tanh\left(\sqrt{32k}\,t\right)$$

by performing the following integration and simplifying the result.

$$\int \frac{dv}{32 - kv^2} = -\int dt$$

(d) Use the result in part (c) to find $\lim_{t\to\infty} v(t)$ and give its interpretation.

(e) Integrate the velocity function in part (c) and find the position s of the object as a function of t. Use a graphing utility to graph the position function when $k = 0.01$ and the position function in part (b) in the same viewing window. Estimate the additional time required for the object to reach ground level when air resistance is not neglected.

(f) Give a written description of what you believe would happen if k were increased. Then test your assertion with a particular value of k.

Tractrix **In Exercises 93 and 94, use the equation of the tractrix** $y = a\,\mathrm{sech}^{-1}\frac{x}{a} - \sqrt{a^2 - x^2},\quad a > 0.$

93. Find dy/dx.

94. Let L be the tangent line to the tractrix at the point P. If L intersects the y-axis at the point Q, show that the distance between P and Q is a.

95. Prove that $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad -1 < x < 1.$

96. Show that $\arctan(\sinh x) = \arcsin(\tanh x)$.

97. Let $x > 0$ and $b > 0$. Show that $\int_{-b}^{b} e^{xt}\,dt = \frac{2\sinh bx}{x}$.

In Exercises 98–102, verify the differentiation formula.

98. $\frac{d}{dx}[\cosh x] = \sinh x$

99. $\frac{d}{dx}[\mathrm{sech}^{-1} x] = \frac{-1}{x\sqrt{1-x^2}}$

100. $\frac{d}{dx}[\cosh^{-1} x] = \frac{1}{\sqrt{x^2-1}}$

101. $\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{x^2+1}}$

102. $\frac{d}{dx}[\mathrm{sech}\, x] = -\mathrm{sech}\, x \tanh x$

Putnam Exam Challenge

103. From the vertex $(0, c)$ of the catenary $y = c\cosh(x/c)$ a line L is drawn perpendicular to the tangent to the catenary at a point P. Prove that the length of L intercepted by the axes is equal to the ordinate y of the point P.

104. Prove or disprove that there is at least one straight line normal to the graph of $y = \cosh x$ at a point $(a, \cosh a)$ and also normal to the graph of $y = \sinh x$ at a point $(c, \sinh c)$.

[At a point on a graph, the normal line is the perpendicular to the tangent at that point. Also, $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$.]

These problems were composed by the Committee on the Putnam Prize Competition.

Section Project: St. Louis Arch

Roy Ooms/Masterfile

The Gateway Arch in St. Louis, Missouri was constructed using the hyperbolic cosine function. The equation used for construction was

$y = 693.8597 - 68.7672\cosh 0.0100333x,$

$-299.2239 \le x \le 299.2239$

where x and y are measured in feet. Cross sections of the arch are equilateral triangles, and (x, y) traces the path of the centers of mass of the cross-sectional triangles. For each value of x, the area of the cross-sectional triangle is $A = 125.1406\cosh 0.0100333x$. (*Source:* Owner's Manual for the Gateway Arch, Saint Louis, MO, by William Thayer)

(a) How high above the ground is the center of the highest triangle? (At ground level, $y = 0$.)

(b) What is the height of the arch? (*Hint:* For an equilateral triangle, $A = \sqrt{3}c^2$, where c is one-half the base of the triangle, and the center of mass of the triangle is located at two-thirds the height of the triangle.)

(c) How wide is the arch at ground level?

Review Exercises for Chapter 5

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1 and 2, sketch the graph of the function by hand. Identify any asymptotes of the graph.

1. $f(x) = \ln x + 3$
2. $f(x) = \ln(x - 3)$

In Exercises 3 and 4, use the properties of logarithms to expand the logarithmic function.

3. $\ln \sqrt[5]{\dfrac{4x^2 - 1}{4x^2 + 1}}$
4. $\ln[(x^2 + 1)(x - 1)]$

In Exercises 5 and 6, write the expression as the logarithm of a single quantity.

5. $\ln 3 + \frac{1}{3}\ln(4 - x^2) - \ln x$
6. $3[\ln x - 2\ln(x^2 + 1)] + 2\ln 5$

In Exercises 7 and 8, solve the equation for x.

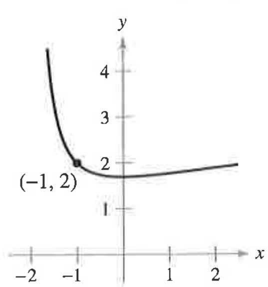
7. $\ln \sqrt{x + 1} = 2$
8. $\ln x + \ln(x - 3) = 0$

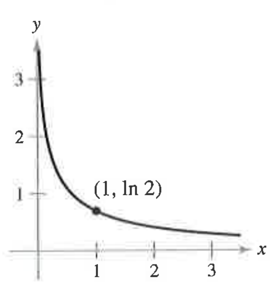
In Exercises 9–14, find the derivative of the function.

9. $g(x) = \ln \sqrt{x}$
10. $h(x) = \ln \dfrac{x(x - 1)}{x - 2}$
11. $f(x) = x\sqrt{\ln x}$
12. $f(x) = \ln[x(x^2 - 2)^{2/3}]$
13. $y = \dfrac{1}{b^2}[a + bx - a\ln(a + bx)]$
14. $y = -\dfrac{1}{ax} + \dfrac{b}{a^2}\ln\dfrac{a + bx}{x}$

In Exercises 15 and 16, find an equation of the tangent line to the graph of the function at the given point.

15. $y = \ln(2 + x) + \dfrac{2}{2 + x}$
16. $y = \ln \dfrac{1 + x}{x}$





In Exercises 17–24, find or evaluate the integral.

17. $\displaystyle\int \frac{1}{7x - 2}\,dx$
18. $\displaystyle\int \frac{x}{x^2 - 1}\,dx$
19. $\displaystyle\int \frac{\sin x}{1 + \cos x}\,dx$
20. $\displaystyle\int \frac{\ln \sqrt{x}}{x}\,dx$
21. $\displaystyle\int_1^4 \frac{x + 1}{x}\,dx$
22. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$
23. $\displaystyle\int_0^{\pi/3} \sec\theta\,d\theta$
24. $\displaystyle\int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right)dx$

In Exercises 25–30, (a) find the inverse function of f, (b) use a graphing utility to graph f and f^{-1} in the same viewing window, and (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

25. $f(x) = \frac{1}{2}x - 3$
26. $f(x) = 5x - 7$
27. $f(x) = \sqrt{x + 1}$
28. $f(x) = x^3 + 2$
29. $f(x) = \sqrt[3]{x + 1}$
30. $f(x) = x^2 - 5, \quad x \ge 0$

In Exercises 31–34, find $(f^{-1})'(a)$ for the function f and the given real number a.

31. $f(x) = x^3 + 2, \quad a = -1$
32. $f(x) = x\sqrt{x - 3}, \quad a = 4$
33. $f(x) = \tan x, \quad -\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}, \quad a = \dfrac{\sqrt{3}}{3}$
34. $f(x) = \ln x, \quad a = 0$

In Exercises 35 and 36, (a) find the inverse function of f, (b) use a graphing utility to graph f and f^{-1} in the same viewing window, and (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

35. $f(x) = \ln \sqrt{x}$
36. $f(x) = e^{1-x}$

In Exercises 37 and 38, graph the function without the aid of a graphing utility.

37. $y = e^{-x/2}$
38. $y = 4e^{-x^2}$

In Exercises 39–44, find the derivative of the function.

39. $g(t) = t^2e^t$
40. $g(x) = \ln \dfrac{e^x}{1 + e^x}$
41. $y = \sqrt{e^{2x} + e^{-2x}}$
42. $h(z) = e^{-z^2/2}$
43. $g(x) = \dfrac{x^2}{e^x}$
44. $y = 3e^{-3/t}$

In Exercises 45 and 46, find an equation of the tangent line to the graph of the function at the given point.

45. $f(x) = \ln(e^{-x^2}), \quad (2, -4)$
46. $f(\theta) = \dfrac{1}{2}e^{\sin 2\theta}, \quad \left(0, \dfrac{1}{2}\right)$

In Exercises 47 and 48, use implicit differentiation to find dy/dx.

47. $y\ln x + y^2 = 0$
48. $\cos x^2 = xe^y$

In Exercises 49–56, find or evaluate the integral.

49. $\displaystyle\int_0^1 xe^{-3x^2}\,dx$
50. $\displaystyle\int_{1/2}^2 \frac{e^{1/x}}{x^2}\,dx$
51. $\displaystyle\int \frac{e^{4x} - e^{2x} + 1}{e^x}\,dx$
52. $\displaystyle\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\,dx$

53. $\int xe^{1-x^2}\,dx$

54. $\int x^2 e^{x^3+1}\,dx$

55. $\int_1^3 \frac{e^x}{e^x - 1}\,dx$

56. $\int_0^2 \frac{e^{2x}}{e^{2x}+1}\,dx$

57. Show that $y = e^x(a\cos 3x + b\sin 3x)$ satisfies the differential equation $y'' - 2y' + 10y = 0$.

58. ***Depreciation*** The value V of an item t years after it is purchased is $V = 8000e^{-0.6t}$, $0 \le t \le 5$.

(a) Use a graphing utility to graph the function.

(b) Find the rates of change of V with respect to t when $t = 1$ and $t = 4$.

(c) Use a graphing utility to graph the tangent lines to the function when $t = 1$ and $t = 4$.

In Exercises 59 and 60, find the area of the region bounded by the graphs of the equations.

59. $y = xe^{-x^2}$, $y = 0$, $x = 0$, $x = 4$

60. $y = 2e^{-x}$, $y = 0$, $x = 0$, $x = 2$

In Exercises 61–64, sketch the graph of the function by hand.

61. $y = 3^{x/2}$

62. $y = 6(2^{-x^2})$

63. $y = \log_2(x - 1)$

64. $y = \log_4 x^2$

In Exercises 65–70, find the derivative of the function.

65. $f(x) = 3^{x-1}$

66. $f(x) = (4e)^x$

67. $y = x^{2x+1}$

68. $y = x(4^{-x})$

69. $g(x) = \log_3 \sqrt{1-x}$

70. $h(x) = \log_5 \frac{x}{x-1}$

In Exercises 71 and 72, find the indefinite integral.

71. $\int (x+1)5^{(x+1)^2}\,dx$

72. $\int \frac{2^{-1/t}}{t^2}\,dt$

73. ***Climb Rate*** The time t (in minutes) for a small plane to climb to an altitude of h feet is

$$t = 50\log_{10}\frac{18{,}000}{18{,}000 - h}$$

where 18,000 feet is the plane's absolute ceiling.

(a) Determine the domain of the function appropriate for the context of the problem.

(b) Use a graphing utility to graph the time function and identify any asymptotes.

(c) Find the time when the altitude is increasing at the greatest rate.

74. ***Compound Interest***

(a) How large a deposit, at 7% interest compounded continuously, must be made to obtain a balance of \$10,000 in 15 years?

(b) A deposit earns interest at a rate of r percent compounded continuously and doubles in value in 10 years. Find r.

In Exercises 75 and 76, sketch the graph of the function.

75. $f(x) = 2\arctan(x+3)$

76. $h(x) = -3\arcsin 2x$

In Exercises 77 and 78, evaluate the expression without using a calculator. (*Hint:* Make a sketch of a right triangle.)

77. (a) $\sin\left(\arcsin \frac{1}{2}\right)$
(b) $\cos\left(\arcsin \frac{1}{2}\right)$

78. (a) $\tan(\operatorname{arccot} 2)$
(b) $\cos\left(\operatorname{arcsec}\sqrt{5}\right)$

In Exercises 79–84, find the derivative of the function.

79. $y = \tan(\arcsin x)$

80. $y = \arctan(x^2 - 1)$

81. $y = x \operatorname{arcsec} x$

82. $y = \frac{1}{2}\arctan e^{2x}$

83. $y = x(\arcsin x)^2 - 2x + 2\sqrt{1-x^2}\arcsin x$

84. $y = \sqrt{x^2-4} - 2\operatorname{arcsec}\frac{x}{2}$, $2 < x < 4$

In Exercises 85–90, find the indefinite integral.

85. $\int \frac{1}{e^{2x}+e^{-2x}}\,dx$

86. $\int \frac{1}{3+25x^2}\,dx$

87. $\int \frac{x}{\sqrt{1-x^4}}\,dx$

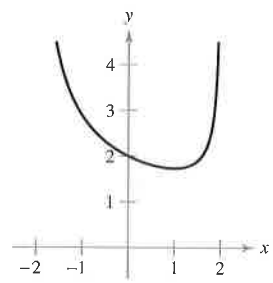
88. $\int \frac{1}{16+x^2}\,dx$

89. $\int \frac{\arctan(x/2)}{4+x^2}\,dx$

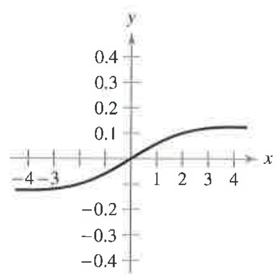
90. $\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

In Exercises 91 and 92, find the area of the region.

91. $y = \frac{4-x}{\sqrt{4-x^2}}$



92. $y = \frac{x}{16+x^2}$



93. ***Harmonic Motion*** A weight of mass m is attached to a spring and oscillates with simple harmonic motion. By Hooke's Law, you can determine that

$$\int \frac{dy}{\sqrt{A^2 - y^2}} = \int \sqrt{\frac{k}{m}}\,dt$$

where A is the maximum displacement, t is the time, and k is a constant. Find y as a function of t, given that $y = 0$ when $t = 0$.

In Exercises 94 and 95, find the derivative of the function.

94. $y = 2x - \cosh\sqrt{x}$

95. $y = x\tanh^{-1} 2x$

In Exercises 96 and 97, find the indefinite integral.

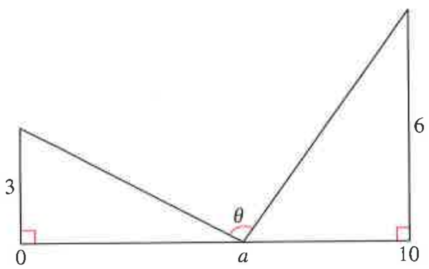
96. $\int \frac{x}{\sqrt{x^4-1}}\,dx$

97. $\int x^2 \operatorname{sech}^2 x^3\,dx$

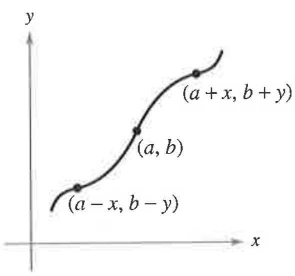
P.S. Problem Solving

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. Find the value of a that maximizes the angle θ shown in the figure. What is the approximate measure of this angle?



2. Recall that the graph of a function $y = f(x)$ is symmetric with respect to the origin if whenever (x, y) is a point on the graph, $(-x, -y)$ is also a point on the graph. The graph of the function $y = f(x)$ is **symmetric with respect to the point (a, b)** if, whenever $(a - x, b - y)$ is a point on the graph, $(a + x, b + y)$ is also a point on the graph, as shown in the figure.



(a) Sketch the graph of $y = \sin x$ on the interval $[0, 2\pi]$. Write a short paragraph explaining how the symmetry of the graph with respect to the point $(0, \pi)$ allows you to conclude that

$$\int_0^{2\pi} \sin x \, dx = 0.$$

(b) Sketch the graph of $y = \sin x + 2$ on the interval $[0, 2\pi]$. Use the symmetry of the graph with respect to the point $(\pi, 2)$ to evaluate the integral

$$\int_0^{2\pi} (\sin x + 2) \, dx.$$

(c) Sketch the graph of $y = \arccos x$ on the interval $[-1, 1]$. Use the symmetry of the graph to evaluate the integral

$$\int_{-1}^{1} \arccos x \, dx.$$

(d) Evaluate the integral $\displaystyle\int_0^{\pi/2} \frac{1}{1 + (\tan x)^{\sqrt{2}}} \, dx.$

3. (a) Use a graphing utility to graph $f(x) = \dfrac{\ln(x + 1)}{x}$ on the interval $[-1, 1]$.

(b) Use the graph to estimate $\lim\limits_{x \to 0} f(x)$.

(c) Use the definition of derivative to prove your answer to part (b).

4. Let $f(x) = \sin(\ln x)$.

(a) Determine the domain of the function f.

(b) Find two values of x satisfying $f(x) = 1$.

(c) Find two values of x satisfying $f(x) = -1$.

(d) What is the range of the function f?

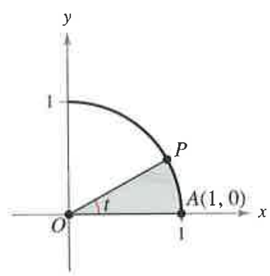
(e) Calculate $f'(x)$ and use calculus to find the maximum value of f on the interval $[1, 10]$.

(f) Use a graphing utility to graph f in the viewing window $[0, 5] \times [-2, 2]$ and estimate $\lim\limits_{x \to 0^+} f(x)$, if it exists.

(g) Determine $\lim\limits_{x \to 0^+} f(x)$ analytically, if it exists.

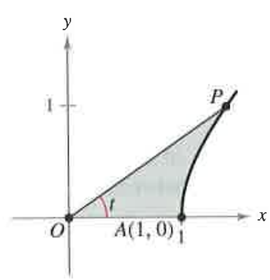
5. Graph the exponential function $y = a^x$ for $a = 0.5, 1.2$, and 2.0. Which of these curves intersects the line $y = x$? Determine all positive numbers a for which the curve $y = a^x$ intersects the line $y = x$.

6. (a) Let $P(\cos t, \sin t)$ be a point on the unit circle $x^2 + y^2 = 1$ in the first quadrant (see figure). Show that t is equal to twice the area of the shaded circular sector AOP.

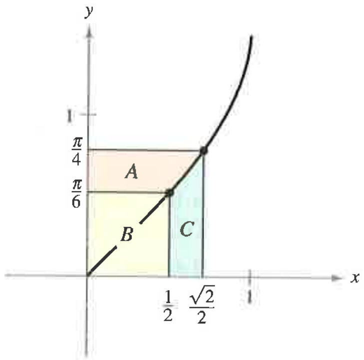


(b) Let $P(\cosh t, \sinh t)$ be a point on the unit hyperbola $x^2 - y^2 = 1$ in the first quadrant (see figure). Show that t is equal to twice the area of the shaded region AOP. Begin by showing that the area of the shaded region AOP is given by the formula

$$A(t) = \frac{1}{2} \cosh t \sinh t - \int_1^{\cosh t} \sqrt{x^2 - 1} \, dx.$$



7. Consider the three regions A, B, and C determined by the graph of $f(x) = \arcsin x$, as shown in the figure.



(a) Calculate the areas of regions A and B.

(b) Use your answers in part (a) to evaluate the integral

$$\int_{1/2}^{\sqrt{2}/2} \arcsin x \, dx.$$

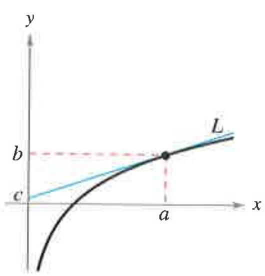
(c) Use your answers in part (a) to evaluate the integral

$$\int_{1}^{3} \ln x \, dx.$$

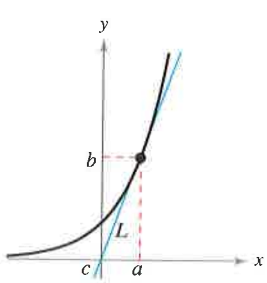
(d) Use your answers in part (a) to evaluate the integral

$$\int_{1}^{\sqrt{3}} \arctan x \, dx.$$

8. Let L be the tangent line to the graph of the function $y = \ln x$ at the point (a, b). Show that the distance between b and c is always equal to 1.



9. Let L be the tangent line to the graph of the function $y = e^x$ at the point (a, b). Show that the distance between a and c is always equal to 1.



10. Use integration by substitution to find the area under the curve

$$y = \frac{1}{\sqrt{x} + x}$$

between $x = 1$ and $x = 4$.

11. Use integration by substitution to find the area under the curve

$$y = \frac{1}{\sin^2 x + 4\cos^2 x}$$

between $x = 0$ and $x = \pi/4$.

12. (a) Use a graphing utility to compare the graph of the function $y = e^x$ with the graphs of each of the given functions.

(i) $y_1 = 1 + \dfrac{x}{1!}$

(ii) $y_2 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!}$

(iii) $y_3 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$

(b) Identify the pattern of successive polynomials in part (a) and extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = e^x$.

(c) What do you think this pattern implies?

13. A \$120,000 home mortgage for 35 years at $9\frac{1}{2}\%$ has a monthly payment of \$985.93. Part of the monthly payment goes for the interest charge on the unpaid balance and the remainder of the payment is used to reduce the principal. The amount that goes for interest is

$$u = M - \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}$$

and the amount that goes toward reduction of the principal is

$$v = \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}.$$

In these formulas, P is the amount of the mortgage, r is the interest rate, M is the monthly payment, and t is the time in years.

(a) Use a graphing utility to graph each function in the same viewing window. (The viewing window should show all 35 years of mortgage payments.)

(b) In the early years of the mortgage, the larger part of the monthly payment goes for what purpose? Approximate the time when the monthly payment is evenly divided between interest and principal reduction.

(c) Use the graphs in part (a) to make a conjecture about the relationship between the slopes of the tangent lines to the two curves for a specified value of t. Give an analytical argument to verify your conjecture. Find $u'(15)$ and $v'(15)$.

(d) Repeat parts (a) and (b) for a repayment period of 20 years ($M = \$1118.56$). What can you conclude?