Limits and Their Properties

Properties of Limits:

If L, M, c, and k are real numbers and $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

1. Sum Rule: $\lim_{x\to c}\left(f(x)+g(x)\right) = L + M$
2. Difference Rule: $\lim_{x\to c}\left(f(x)-g(x)\right) = L - M$
3. Product Rule: $\lim_{x\to c}\left(f(x)\cdot g(x)\right) = L \cdot M$
4. Quotient Rule: $\lim_{x\to c}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{L}{M}$ where $M \neq 0$
5. Constant Multiple Rule: $\lim_{x\to c}\left(k \cdot f(x)\right) = k \cdot L$
6. Power Rule: If r and s are integers, $s \neq 0$, then $\lim_{x\to c}\left(f(x)\right)^{r/s} = L^{r/s}$ provided that $L^{r/s}$ is a real number.
7. Limit of a Composite Function Rule: If f and g are functions such that $\lim_{x\to c} g(x) = L$ and $\lim_{x\to c} f(x) = f(L)$, then $\lim_{x\to c} f\left(g(x)\right) = f\left(\lim_{x\to c} g(x)\right) = f(L)$.

Ex. Given that $\lim_{x\to a} f(x) = 2$ and $\lim_{x\to a} g(x) = 3$, find the limit if it exists.

(a) $\lim_{x\to a}\left(5g(x)\right) =$

(b) $\lim_{x\to a} \dfrac{f(x)}{g(x)} =$

Two special trig limits:

$$\boxed{\lim_{\theta\to 0} \frac{\sin\theta}{\theta} =}$$

$$\boxed{\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} =}$$

Ex. $\lim_{x\to 0} \dfrac{\sin(5x)}{4x} =$

Ex. $\lim_{x\to 0} \dfrac{2x+\sin x}{x} =$

Squeeze Theorem or Sandwich Theorem:

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possible at c itself, and if

$\lim_{x \to c} h(x) = L$ and $\lim_{x \to c} g(x) = L$, then $\lim_{x \to c} f(x) = L$.

Ex. If $2 \le f(x) \le x^2 + 2$ for all x, find $\lim_{x \to 0} f(x)$.

Continuity

Definition of Continuity

A function f is said to be continuous at $x = c$ if and only if:

1)

2)

3)

Ex. Sketch a function f so that:

(a) $f(c)$ is not defined

(b) $\lim_{x \to c} f(x)$ does not exist

(c) $f(c)$ is defined and $\lim_{x \to c} f(x)$ exists but $\lim_{x \to c} f(x) \neq f(c)$

(d) f is continous at $x = c$

If a function f is not continuous at $x = c$, the discontinuity may be one of three types:

1) point discontinuity	2) jump discontinuity	3) asymptotic discontinuity

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A point discontinuity is said to be a **removable discontinuity** because the function can be redefined at the point in such as way as to make the function continuous there. Jump discontinuities and asymptotic discontinuities are non-removable because the functions which contain them cannot be redefined at a point to make them continuous.

Ex. 1) Find the value(s) of x at which the given function is discontinuous.

2) Identify each value as a point, jump, or asymptotic discontinuity

3) Identify each value as a removable or non-removable discontinuity. If it is removable, redefine the function at that value so that it will be continuous.

4) Sketch the graph.

(a) $f(x)=\dfrac{x^2-x-6}{x-3}$

(b) $f(x)=\dfrac{1}{x-3}$

(c) $f(x)=\begin{cases} x+2 \text{ if } x<1 \\ 2-x \text{ if } x>1 \end{cases}$

Ex. Find k so that f will be continuous at $x=2$ given $f(x)=\begin{cases} x+3 \text{ if } x\le 2 \\ kx+6 \text{ if } x>2 \end{cases}$. Sketch f.

Homework: P 79: 25, 26, 32, 33, 34, 37, 45, 57, 58, 60

Remember that the work for the odd-numbered problems can be found on www.calcchat.com

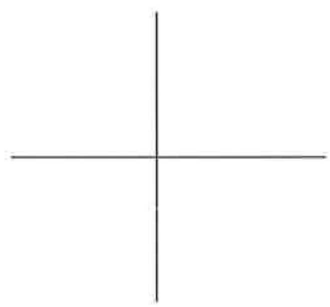
Intermediate Value Theorem
Since a person's height is a continuous function of time, if a child had a height of 58 inches at age 11 and a height of 64 inches at age 14, then the child's height took on every value between 58 inches and 64 inches for some time between the age of 11 and the age of 14. This is a simple example of an important theorem in Calculus called the Intermediate Value Theorem.

Intermediate Value Theorem
If: i) $f(x)$ is continuous on the closed interval $[a, b]$
ii) if $f(a) \neq f(b)$
iii) if k is between $f(a)$ and $f(b)$,
then there exists a number c between a and b for which $f(c) = k$.

In other words, a function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$.
If k is between $f(a)$ and $f(b)$, then there is at least one value c in (a, b) for which $f(c) = k$.

Ex. (a) Determine if the Intermediate Value Theorem holds for the given value of k.
(b) If the theorem holds, find a number c for which $f(c) = k$. If the theorem does not hold, give the reason.
(c) Draw a sketch of the curve and the line $y = k$.

1) $f(x) = \dfrac{1}{x-2}$, $[a, b] = \left[2\frac{1}{2}, 7\right]$, $k = \dfrac{1}{4}$



2) $f(x) = x^2 + 5x - 6$, $[a, b] = [-1, 2]$, $k = 4$

Homework: Worksheet