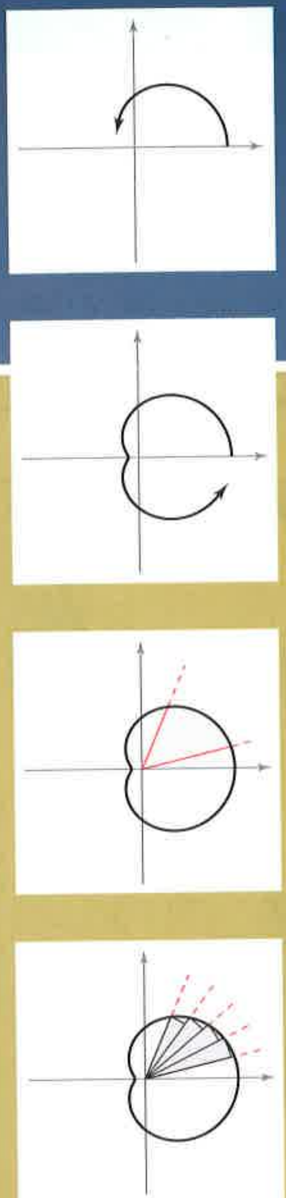
10 Conics, Parametric Equations, and Polar Coordinates

During the 2002 Winter Olympic Games, the Olympic rings were lighted high on a mountainside in Salt Lake City. The volunteers who installed the lights for the display took care to minimize the environmental impact of the project. How can you calculate the area enclosed by the display? Explain.



Harry Howe/Getty Images

In the polar coordinate system, graphing an equation involves tracing a curve about a fixed point called the pole. Consider a region bounded by a curve and by the rays that contain the endpoints of an interval on the curve. You can use sectors of circles to approximate the area of such a region. In Chapter 10, you will see how the limit process can be used to find this area.

Section 10.1

Conics and Calculus

- **Understand the definition of a conic section.**
- **Analyze and write equations of parabolas using properties of parabolas.**
- **Analyze and write equations of ellipses using properties of ellipses.**
- **Analyze and write equations of hyperbolas using properties of hyperbolas.**

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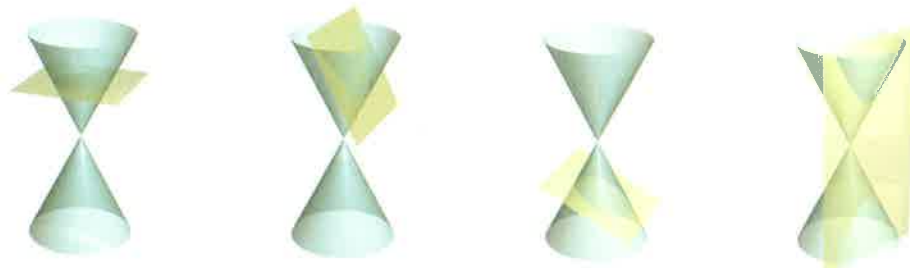
HYPATIA (370–415 A.D.)

The Greeks discovered conic sections sometime between 600 and 300 B.C. By the beginning of the Alexandrian period, enough was known about conics for Apollonius (262–190 B.C.) to produce an eight-volume work on the subject. Later, toward the end of the Alexandrian period, Hypatia wrote a textbook entitled *On the Conics of Apollonius*. Her death marked the end of major mathematical discoveries in Europe for several hundred years.

The early Greeks were largely concerned with the geometric properties of conics. It was not until 1900 years later, in the early seventeenth century, that the broader applicability of conics became apparent. Conics then played a prominent role in the development of calculus.

Conic Sections

Each **conic section** (or simply **conic**) can be described as the intersection of a plane and a double-napped cone. Notice in Figure 10.1 that for the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane passes through the vertex, the resulting figure is a **degenerate conic,** as shown in Figure 10.2.



Circle Parabola Ellipse Hyperbola

Conic sections
Figure 10.1

Point Line Two intersecting lines

Degenerate conics
Figure 10.2

There are several ways to study conics. You could begin as the Greeks did by defining the conics in terms of the intersections of planes and cones, or you could define them algebraically in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \qquad \text{General second-degree equation}$$

However, a third approach, in which each of the conics is defined as a **locus** (collection) of points satisfying a certain geometric property, works best. For example, a circle can be defined as the collection of all points (x, y) that are equidistant from a fixed point (h, k). This locus definition easily produces the standard equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2. \qquad \text{Standard equation of a circle}$$

FOR FURTHER INFORMATION To learn more about the mathematical activities of Hypatia, see the article "Hypatia and Her Mathematics" by Michael A. B. Deakin in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

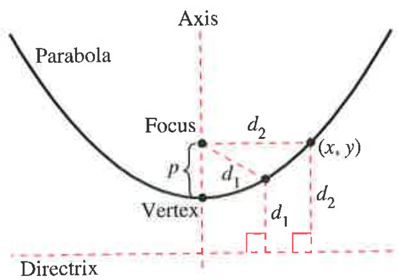


Figure 10.3

Parabolas

A **parabola** is the set of all points (x, y) that are equidistant from a fixed line called the **directrix** and a fixed point called the **focus** not on the line. The midpoint between the focus and the directrix is the **vertex,** and the line passing through the focus and the vertex is the **axis** of the parabola. Note in Figure 10.3 that a parabola is symmetric with respect to its axis.

THEOREM 10.1 Standard Equation of a Parabola

The **standard form** of the equation of a parabola with vertex (h, k) and directrix $y = k - p$ is

$(x - h)^2 = 4p(y - k)$. Vertical axis

For directrix $x = h - p$, the equation is

$(y - k)^2 = 4p(x - h)$. Horizontal axis

The focus lies on the axis p units (*directed distance*) from the vertex. The coordinates of the focus are as follows.

$(h, k + p)$ Vertical axis

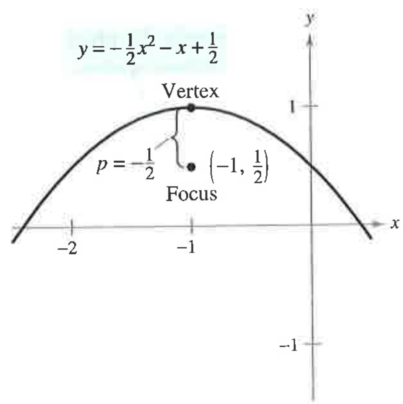
$(h + p, k)$ Horizontal axis

EXAMPLE 1 Finding the Focus of a Parabola

Find the focus of the parabola given by $y = -\frac{1}{2}x^2 - x + \frac{1}{2}$.

Solution To find the focus, convert to standard form by completing the square.

$$
\begin{aligned}
y &= \tfrac{1}{2} - x - \tfrac{1}{2}x^2 && \text{Rewrite original equation.}\\
y &= \tfrac{1}{2}(1 - 2x - x^2) && \text{Factor out } \tfrac{1}{2}.\\
2y &= 1 - 2x - x^2 && \text{Multiply each side by 2.}\\
2y &= 1 - (x^2 + 2x) && \text{Group terms.}\\
2y &= 2 - (x^2 + 2x + 1) && \text{Add and subtract 1 on right side.}\\
x^2 + 2x + 1 &= -2y + 2 &&\\
(x + 1)^2 &= -2(y - 1) && \text{Write in standard form.}
\end{aligned}
$$



Parabola with a vertical axis, $p < 0$
Figure 10.4

Comparing this equation with $(x - h)^2 = 4p(y - k)$, you can conclude that

$$h = -1, \qquad k = 1, \qquad \text{and} \qquad p = -\tfrac{1}{2}.$$

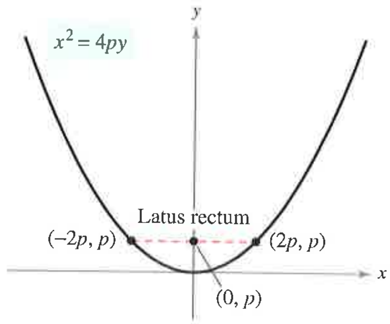
Because p is negative, the parabola opens downward, as shown in Figure 10.4. So, the focus of the parabola is p units from the vertex, or

$(h, k + p) = \left(-1, \frac{1}{2}\right)$. Focus

A line segment that passes through the focus of a parabola and has endpoints on the parabola is called a **focal chord.** The specific focal chord perpendicular to the axis of the parabola is the **latus rectum.** The next example shows how to determine the length of the latus rectum and the length of the corresponding intercepted arc.

EXAMPLE 2 Focal Chord Length and Arc Length

Find the length of the latus rectum of the parabola given by $x^2 = 4py$. Then find the length of the parabolic arc intercepted by the latus rectum.



Length of latus rectum: $4p$
Figure 10.5

Solution Because the latus rectum passes through the focus $(0, p)$ and is perpendicular to the y-axis, the coordinates of its endpoints are $(-x, p)$ and (x, p). Substituting p for y in the equation of the parabola produces

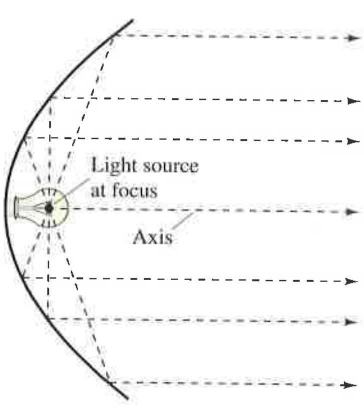
$$x^2 = 4p(p) \quad \Rightarrow \quad x = \pm 2p.$$

So, the endpoints of the latus rectum are $(-2p, p)$ and $(2p, p)$, and you can conclude that its length is $4p$, as shown in Figure 10.5. In contrast, the length of the intercepted arc is

$$\begin{aligned}
s &= \int_{-2p}^{2p} \sqrt{1 + (y')^2}\, dx && \text{Use arc length formula.} \\
&= 2\int_0^{2p} \sqrt{1 + \left(\frac{x}{2p}\right)^2}\, dx && y = \frac{x^2}{4p} \Rightarrow y' = \frac{x}{2p} \\
&= \frac{1}{p}\int_0^{2p} \sqrt{4p^2 + x^2}\, dx && \text{Simplify.} \\
&= \frac{1}{2p}\left[x\sqrt{4p^2 + x^2} + 4p^2 \ln\left|x + \sqrt{4p^2 + x^2}\right|\right]_0^{2p} && \text{Theorem 8.2} \\
&= \frac{1}{2p}\left[2p\sqrt{8p^2} + 4p^2 \ln\left(2p + \sqrt{8p^2}\right) - 4p^2 \ln(2p)\right] \\
&= 2p\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right] \\
&\approx 4.59p.
\end{aligned}$$

One widely used property of a parabola is its reflective property. In physics, a surface is called **reflective** if the tangent line at any point on the surface makes equal angles with an incoming ray and the resulting outgoing ray. The angle corresponding to the incoming ray is the **angle of incidence,** and the angle corresponding to the outgoing ray is the **angle of reflection.** One example of a reflective surface is a flat mirror.

Another type of reflective surface is that formed by revolving a parabola about its axis. A special property of parabolic reflectors is that they allow us to direct all incoming rays parallel to the axis through the focus of the parabola—this is the principle behind the design of the parabolic mirrors used in reflecting telescopes. Conversely, all light rays emanating from the focus of a parabolic reflector used in a flashlight are parallel, as shown in Figure 10.6.



Parabolic reflector: light is reflected in parallel rays.
Figure 10.6

THEOREM 10.2 Reflective Property of a Parabola

Let P be a point on a parabola. The tangent line to the parabola at the point P makes equal angles with the following two lines.

1. The line passing through P and the focus
2. The line passing through P parallel to the axis of the parabola

indicates that in the HM mathSpace® *CD-ROM and the online* Eduspace® *system for this text, you will find an Open Exploration, which further explores this example using the computer algebra systems* Maple, Mathcad, Mathematica, *and* Derive.

Bettmann/Corbis

NICOLAUS COPERNICUS (1473–1543)

Copernicus began to study planetary motion when asked to revise the calendar. At that time, the exact length of the year could not be accurately predicted using the theory that Earth was the center of the universe.

Ellipses

More than a thousand years after the close of the Alexandrian period of Greek mathematics, Western civilization finally began a Renaissance of mathematical and scientific discovery. One of the principal figures in this rebirth was the Polish astronomer Nicolaus Copernicus. In his work *On the Revolutions of the Heavenly Spheres*, Copernicus claimed that all of the planets, including Earth, revolved about the sun in circular orbits. Although some of Copernicus's claims were invalid, the controversy set off by his heliocentric theory motivated astronomers to search for a mathematical model to explain the observed movements of the sun and planets. The first to find an accurate model was the German astronomer Johannes Kepler (1571–1630). Kepler discovered that the planets move about the sun in elliptical orbits, with the sun not as the center but as a focal point of the orbit.

The use of ellipses to explain the movements of the planets is only one of many practical and aesthetic uses. As with parabolas, you will begin your study of this second type of conic by defining it as a locus of points. Now, however, *two* focal points are used rather than one.

An **ellipse** is the set of all points (x, y) the sum of whose distances from two distinct fixed points called **foci** is constant. (See Figure 10.7.) The line through the foci intersects the ellipse at two points, called the **vertices.** The chord joining the vertices is the **major axis,** and its midpoint is the **center** of the ellipse. The chord perpendicular to the major axis at the center is the **minor axis** of the ellipse. (See Figure 10.8.)

FOR FURTHER INFORMATION To learn about how an ellipse may be "exploded" into a parabola, see the article "Exploding the Ellipse" by Arnold Good in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com.*

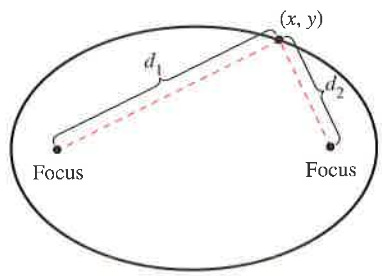


Figure 10.7

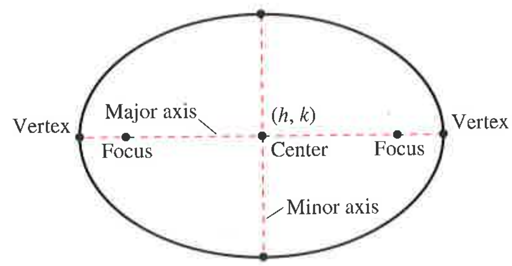


Figure 10.8

THEOREM 10.3 Standard Equation of an Ellipse

The standard form of the equation of an ellipse with center (h, k) and major and minor axes of lengths $2a$ and $2b$, where $a > b$, is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad \text{Major axis is horizontal.}$$

or

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1. \qquad \text{Major axis is vertical.}$$

The foci lie on the major axis, c units from the center, with $c^2 = a^2 - b^2$.

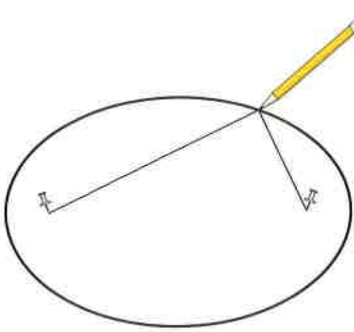


Figure 10.9

NOTE You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure 10.9. If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, the path traced by the pencil will be an ellipse.

EXAMPLE 3 Completing the Square

Find the center, vertices, and foci of the ellipse given by

$$4x^2 + y^2 - 8x + 4y - 8 = 0.$$

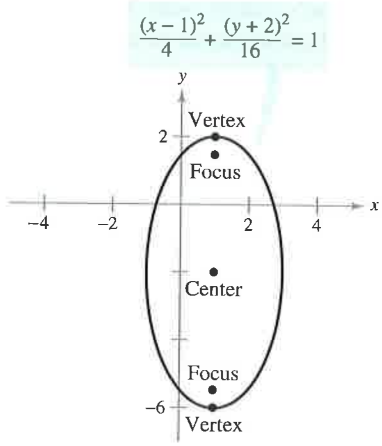
Solution By completing the square, you can write the original equation in standard form.

$$4x^2 + y^2 - 8x + 4y - 8 = 0 \quad \text{Write original equation.}$$
$$4x^2 - 8x + y^2 + 4y = 8$$
$$4(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 4 + 4$$
$$4(x-1)^2 + (y+2)^2 = 16$$
$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1 \quad \text{Write in standard form.}$$

So, the major axis is parallel to the y-axis, where $h = 1$, $k = -2$, $a = 4$, $b = 2$, and $c = \sqrt{16-4} = 2\sqrt{3}$. So, you obtain the following.

Center: $(1, -2)$ (h, k)
Vertices: $(1, -6)$ and $(1, 2)$ $(h, k \pm a)$
Foci: $(1, -2 - 2\sqrt{3})$ and $(1, -2 + 2\sqrt{3})$ $(h, k \pm c)$

The graph of the ellipse is shown in Figure 10.10.



Ellipse with a vertical major axis
Figure 10.10

NOTE If the constant term $F = -8$ in the equation in Example 3 had been greater than or equal to 8, you would have obtained one of the following degenerate cases.

1. $F = 8$, single point, $(1, -2)$: $\dfrac{(x-1)^2}{4} + \dfrac{(y+2)^2}{16} = 0$
2. $F > 8$, no solution points: $\dfrac{(x-1)^2}{4} + \dfrac{(y+2)^2}{16} < 0$

EXAMPLE 4 The Orbit of the Moon

The moon orbits Earth in an elliptical path with the center of Earth at one focus, as shown in Figure 10.11. The major and minor axes of the orbit have lengths of 768,800 kilometers and 767,640 kilometers, respectively. Find the greatest and least distances (the apogee and perigee) from Earth's center to the moon's center.

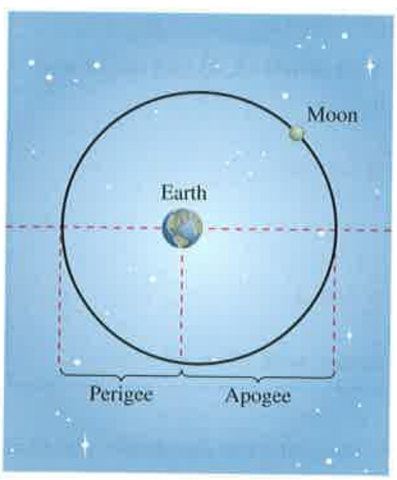


Figure 10.11

Solution Begin by solving for a and b.

$$2a = 768{,}800 \quad \text{Length of major axis}$$
$$a = 384{,}400 \quad \text{Solve for } a.$$
$$2b = 767{,}640 \quad \text{Length of minor axis}$$
$$b = 383{,}820 \quad \text{Solve for } b.$$

Now, using these values, you can solve for c as follows.

$$c = \sqrt{a^2 - b^2} \approx 21{,}108$$

The greatest distance between the center of Earth and the center of the moon is $a + c \approx 405{,}508$ kilometers, and the least distance is $a - c \approx 363{,}292$ kilometers.

FOR FURTHER INFORMATION For more information on some uses of the reflective properties of conics, see the article "Parabolic Mirrors, Elliptic and Hyperbolic Lenses" by Mohsen Maesumi in *The American Mathematical Monthly*. Also see the article "The Geometry of Microwave Antennas" by William R. Parzynski in *Mathematics Teacher*.

Theorem 10.2 presented a reflective property of parabolas. Ellipses have a similar reflective property. You are asked to prove the following theorem in Exercise 110.

THEOREM 10.4 Reflective Property of an Ellipse

Let P be a point on an ellipse. The tangent line to the ellipse at point P makes equal angles with the lines through P and the foci.

One of the reasons that astronomers had difficulty in detecting that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to the center of the sun, making the orbits nearly circular. To measure the ovalness of an ellipse, you can use the concept of **eccentricity.**

Definition of Eccentricity of an Ellipse

The **eccentricity** e of an ellipse is given by the ratio

$$e = \frac{c}{a}.$$

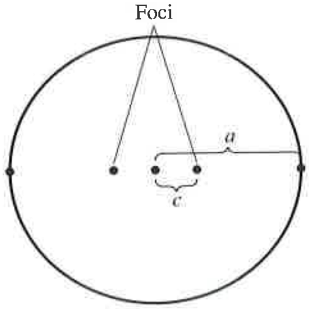
To see how this ratio is used to describe the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that

$$0 < c < a.$$

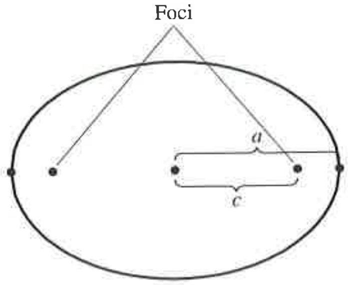
For an ellipse that is nearly circular, the foci are close to the center and the ratio c/a is small, and for an elongated ellipse, the foci are close to the vertices and the ratio is close to 1, as shown in Figure 10.12. Note that $0 < e < 1$ for every ellipse.

The orbit of the moon has an eccentricity of $e = 0.0549$, and the eccentricities of the nine planetary orbits are as follows.

Mercury:	$e = 0.2056$	Saturn:	$e = 0.0542$
Venus:	$e = 0.0068$	Uranus:	$e = 0.0472$
Earth:	$e = 0.0167$	Neptune:	$e = 0.0086$
Mars:	$e = 0.0934$	Pluto:	$e = 0.2488$
Jupiter:	$e = 0.0484$		



(a) $\frac{c}{a}$ is small.



(b) $\frac{c}{a}$ is close to 1.

Eccentricity is the ratio $\frac{c}{a}$.
Figure 10.12

You can use integration to show that the area of an ellipse is $A = \pi ab$. For instance, the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is given by

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\int_0^{\pi/2} a^2 \cos^2\theta\,d\theta. \qquad \text{Trigonometric substitution } x = a\sin\theta.$$

However, it is not so simple to find the *circumference* of an ellipse. The next example shows how to use eccentricity to set up an "elliptic integral" for the circumference of an ellipse.

EXAMPLE 5 Finding the Circumference of an Ellipse

Show that the circumference of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ is

$$4a\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta. \qquad e = \frac{c}{a}$$

Solution Because the given ellipse is symmetric with respect to both the x-axis and the y-axis, you know that its circumference C is four times the arc length of $y = (b/a)\sqrt{a^2 - x^2}$ in the first quadrant. The function y is differentiable for all x in the interval $[0, a]$ except at $x = a$. So, the circumference is given by the improper integral

$$C = \lim_{d \to a} 4\int_0^d \sqrt{1 + (y')^2}\, dx = 4\int_0^a \sqrt{1 + (y')^2}\, dx = 4\int_0^a \sqrt{1 + \frac{b^2x^2}{a^2(a^2 - x^2)}}\, dx.$$

Using the trigonometric substitution $x = a \sin \theta$, you obtain

$$\begin{aligned} C &= 4\int_0^{\pi/2} \sqrt{1 + \frac{b^2 \sin^2 \theta}{a^2 \cos^2 \theta}}\,(a \cos \theta)\, d\theta \\ &= 4\int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}\, d\theta \\ &= 4\int_0^{\pi/2} \sqrt{a^2(1 - \sin^2 \theta) + b^2 \sin^2 \theta}\, d\theta \\ &= 4\int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2)\sin^2 \theta}\, d\theta. \end{aligned}$$

Because $e^2 = c^2/a^2 = (a^2 - b^2)/a^2$, you can rewrite this integral as

$$C = 4a\int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta.$$

AREA AND CIRCUMFERENCE OF AN ELLIPSE

In his work with elliptic orbits in the early 1600's, Johannes Kepler successfully developed a formula for the area of an ellipse, $A = \pi ab$. He was less successful in developing a formula for the circumference of an ellipse, however; the best he could do was to give the approximate formula $C = \pi(a + b)$.

A great deal of time has been devoted to the study of elliptic integrals. Such integrals generally do not have elementary antiderivatives. To find the circumference of an ellipse, you must usually resort to an approximation technique.

EXAMPLE 6 Approximating the Value of an Elliptic Integral

Use the elliptic integral in Example 5 to approximate the circumference of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

Solution Because $e^2 = c^2/a^2 = (a^2 - b^2)/a^2 = 9/25$, you have

$$C = (4)(5)\int_0^{\pi/2} \sqrt{1 - \frac{9 \sin^2 \theta}{25}}\, d\theta.$$

Applying Simpson's Rule with $n = 4$ produces

$$\begin{aligned} C &\approx 20\left(\frac{\pi}{6}\right)\left(\frac{1}{4}\right)[1 + 4(0.9733) + 2(0.9055) + 4(0.8323) + 0.8] \\ &\approx 28.36. \end{aligned}$$

So, the ellipse has a circumference of about 28.36 units, as shown in Figure 10.13.

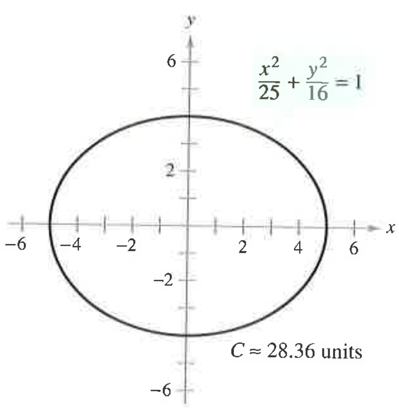


Figure 10.13

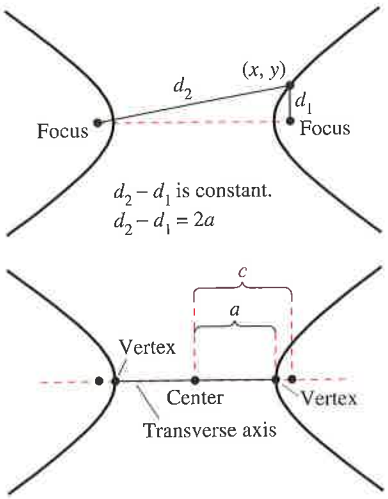


Figure 10.14

Hyperbolas

The definition of a hyperbola is similar to that of an ellipse. For an ellipse, the *sum* of the distances between the foci and a point on the ellipse is fixed, whereas for a hyperbola, the absolute value of the *difference* between these distances is fixed.

A **hyperbola** is the set of all points (x, y) for which the absolute value of the difference between the distances from two distinct fixed points called **foci** is constant. (See Figure 10.14.) The line through the two foci intersects a hyperbola at two points called the **vertices.** The line segment connecting the vertices is the **transverse axis,** and the midpoint of the transverse axis is the **center** of the hyperbola. One distinguishing feature of a hyperbola is that its graph has two separate *branches*.

THEOREM 10.5 Standard Equation of a Hyperbola

The standard form of the equation of a hyperbola with center at (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

or

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \qquad \text{Transverse axis is vertical.}$$

The vertices are a units from the center, and the foci are c units from the center, where, $c^2 = a^2 + b^2$.

NOTE The constants a, b, and c do not have the same relationship for hyperbolas as they do for ellipses. For hyperbolas, $c^2 = a^2 + b^2$, but for ellipses, $c^2 = a^2 - b^2$.

An important aid in sketching the graph of a hyperbola is the determination of its **asymptotes,** as shown in Figure 10.15. Each hyperbola has two asymptotes that intersect at the center of the hyperbola. The asymptotes pass through the vertices of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k). The line segment of length $2b$ joining $(h, k + b)$ and $(h, k - b)$ is referred to as the **conjugate axis** of the hyperbola.

THEOREM 10.6 Asymptotes of a Hyperbola

For a *horizontal* transverse axis, the equations of the asymptotes are

$$y = k + \frac{b}{a}(x - h) \qquad \text{and} \qquad y = k - \frac{b}{a}(x - h).$$

For a *vertical* transverse axis, the equations of the asymptotes are

$$y = k + \frac{a}{b}(x - h) \qquad \text{and} \qquad y = k - \frac{a}{b}(x - h).$$

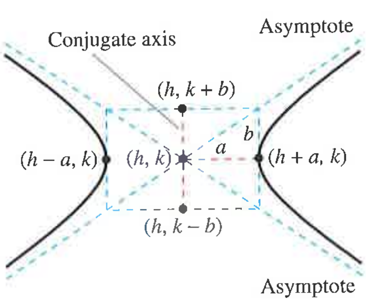


Figure 10.15

In Figure 10.15 you can see that the asymptotes coincide with the diagonals of the rectangle with dimensions $2a$ and $2b$, centered at (h, k). This provides you with a quick means of sketching the asymptotes, which in turn aids in sketching the hyperbola.

EXAMPLE 7 Using Asymptotes to Sketch a Hyperbola

Sketch the graph of the hyperbola whose equation is $4x^2 - y^2 = 16$.

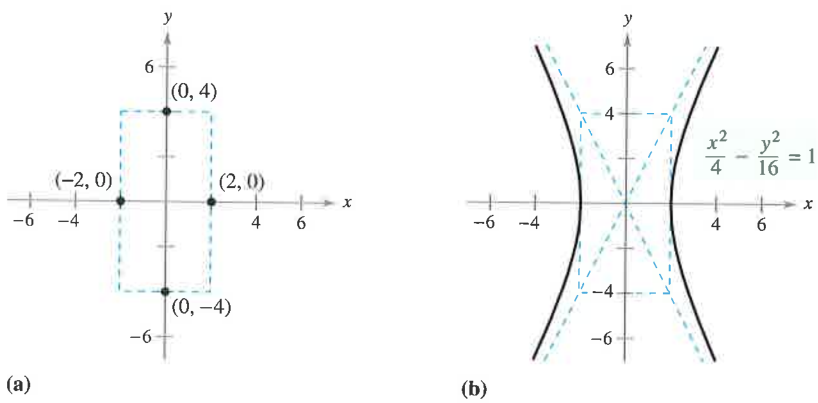
TECHNOLOGY You can use a graphing utility to verify the graph obtained in Example 7 by solving the original equation for y and graphing the following equations.

$$y_1 = \sqrt{4x^2 - 16}$$
$$y_2 = -\sqrt{4x^2 - 16}$$

Solution Begin by rewriting the equation in standard form.

$$\frac{x^2}{4} - \frac{y^2}{16} = 1$$

The transverse axis is horizontal and the vertices occur at $(-2, 0)$ and $(2, 0)$. The ends of the conjugate axis occur at $(0, -4)$ and $(0, 4)$. Using these four points, you can sketch the rectangle shown in Figure 10.16(a). By drawing the asymptotes through the corners of this rectangle, you can complete the sketch as shown in Figure 10.16(b).



(a) (b)

Figure 10.16

Definition of Eccentricity of a Hyperbola

The **eccentricity** e of a hyperbola is given by the ratio

$$e = \frac{c}{a}.$$

As with an ellipse, the **eccentricity** of a hyperbola is $e = c/a$. Because $c > a$ for hyperbolas, it follows that $e > 1$ for hyperbolas. If the eccentricity is large, the branches of the hyperbola are nearly flat. If the eccentricity is close to 1, the branches of the hyperbola are more pointed, as shown in Figure 10.17.

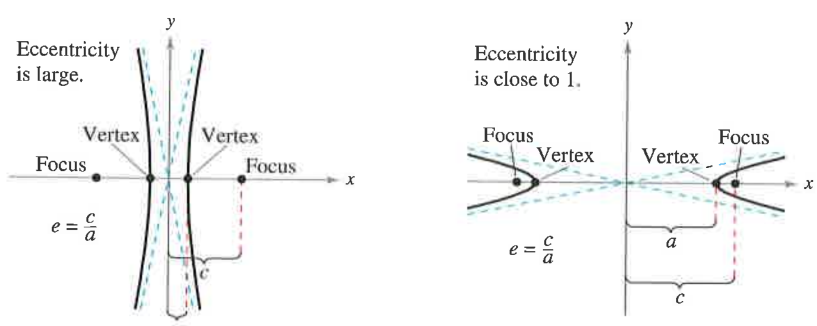
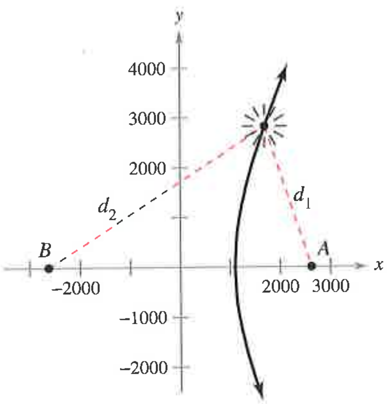


Figure 10.17

The following application was developed during World War II. It shows how the properties of hyperbolas can be used in radar and other detection systems.

EXAMPLE 8 A Hyperbolic Detection System

Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B. Where was the explosion?



$2c = 5280$
$d_2 - d_1 = 2a = 2200$
Figure 10.18

Solution Assuming that sound travels at 1100 feet per second, you know that the explosion took place 2200 feet farther from B than from A, as shown in Figure 10.18. The locus of all points that are 2200 feet closer to A than to B is one branch of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$, where

$$c = \frac{1 \text{ mile}}{2} = \frac{5280 \text{ ft}}{2} = 2640 \text{ feet}$$

and

$$a = \frac{2200 \text{ ft}}{2} = 1100 \text{ feet.}$$

Because $c^2 = a^2 + b^2$, it follows that

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 5{,}759{,}600 \end{aligned}$$

and you can conclude that the explosion occurred somewhere on the right branch of the hyperbola given by

$$\frac{x^2}{1{,}210{,}000} - \frac{y^2}{5{,}759{,}600} = 1.$$

In Example 8, you were able to determine only the hyperbola on which the explosion occurred, but not the exact location of the explosion. If, however, you had received the sound at a third position C, then two other hyperbolas would be determined. The exact location of the explosion would be the point at which these three hyperbolas intersect.

Another interesting application of conics involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. The center of the sun is a focus of each orbit, and each orbit has a vertex at the point at which the comet is closest to the sun. Undoubtedly, many comets with parabolic or hyperbolic orbits have not been identified—such comets pass through our solar system only once. Only comets with elliptical orbits such as Halley's comet remain in our solar system.

The type of orbit for a comet can be determined as follows.

1. Ellipse: $v < \sqrt{2GM/p}$
2. Parabola: $v = \sqrt{2GM/p}$
3. Hyperbola: $v > \sqrt{2GM/p}$

In these three formulas, p is the distance between one vertex and one focus of the comet's orbit (in meters), v is the velocity of the comet at the vertex (in meters per second), $M \approx 1.989 \times 10^{30}$ kilograms is the mass of the sun, and $G \approx 6.67 \times 10^{-8}$ cubic meters per kilogram-second squared is the gravitational constant.

Mary Evans Picture Library

CAROLINE HERSCHEL (1750–1848)

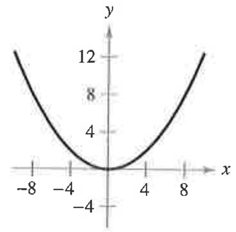
The first woman to be credited with detecting a new comet was the English astronomer Caroline Herschel. During her life, Caroline Herschel discovered a total of eight new comets.

Exercises for Section 10.1

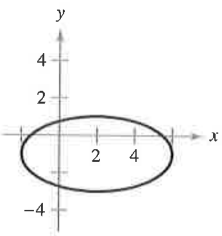
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–8, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]

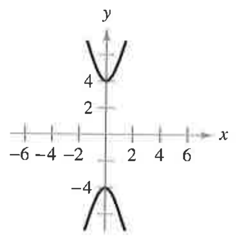
(a)



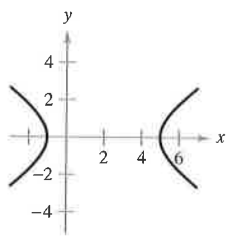
(b)



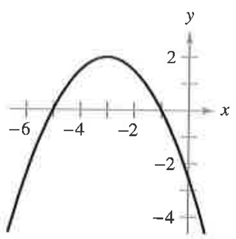
(c)



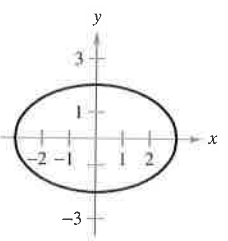
(d)



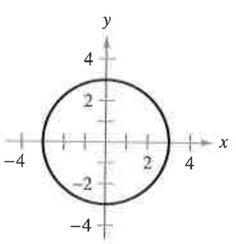
(e)



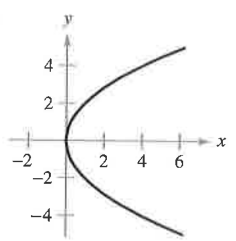
(f)



(g)



(h)



1. $y^2 = 4x$

2. $x^2 = 8y$

3. $(x + 3)^2 = -2(y - 2)$

4. $\dfrac{(x - 2)^2}{16} + \dfrac{(y + 1)^2}{4} = 1$

5. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

6. $\dfrac{x^2}{9} + \dfrac{y^2}{9} = 1$

7. $\dfrac{y^2}{16} - \dfrac{x^2}{1} = 1$

8. $\dfrac{(x - 2)^2}{9} - \dfrac{y^2}{4} = 1$

In Exercises 9–16, find the vertex, focus, and directrix of the parabola, and sketch its graph.

9. $y^2 = -6x$

10. $x^2 + 8y = 0$

11. $(x + 3) + (y - 2)^2 = 0$

12. $(x - 1)^2 + 8(y + 2) = 0$

13. $y^2 - 4y - 4x = 0$

14. $y^2 + 6y + 8x + 25 = 0$

15. $x^2 + 4x + 4y - 4 = 0$

16. $y^2 + 4y + 8x - 12 = 0$

In Exercises 17–20, find the vertex, focus, and directrix of the parabola. Then use a graphing utility to graph the parabola.

17. $y^2 + x + y = 0$

18. $y = -\frac{1}{6}(x^2 - 8x + 6)$

19. $y^2 - 4x - 4 = 0$

20. $x^2 - 2x + 8y + 9 = 0$

In Exercises 21–28, find an equation of the parabola.

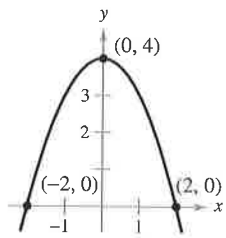
21. Vertex: $(3, 2)$
Focus: $(1, 2)$

22. Vertex: $(-1, 2)$
Focus: $(-1, 0)$

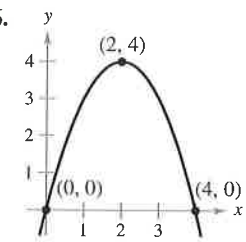
23. Vertex: $(0, 4)$
Directrix: $y = -2$

24. Focus: $(2, 2)$
Directrix: $x = -2$

25.



26.



27. Axis is parallel to y-axis; graph passes through $(0, 3)$, $(3, 4)$, and $(4, 11)$.

28. Directrix: $y = -2$; endpoints of latus rectum are $(0, 2)$ and $(8, 2)$.

In Exercises 29–34, find the center, foci, vertices, and eccentricity of the ellipse, and sketch its graph.

29. $x^2 + 4y^2 = 4$

30. $5x^2 + 7y^2 = 70$

31. $\dfrac{(x - 1)^2}{9} + \dfrac{(y - 5)^2}{25} = 1$

32. $(x + 2)^2 + \dfrac{(y + 4)^2}{1/4} = 1$

33. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

34. $16x^2 + 25y^2 - 64x + 150y + 279 = 0$

In Exercises 35–38, find the center, foci, and vertices of the ellipse. Use a graphing utility to graph the ellipse.

35. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

36. $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

37. $x^2 + 2y^2 - 3x + 4y + 0.25 = 0$

38. $2x^2 + y^2 + 4.8x - 6.4y + 3.12 = 0$

In Exercises 39–44, find an equation of the ellipse.

39. Center: $(0, 0)$
Focus: $(2, 0)$
Vertex: $(3, 0)$

40. Vertices: $(0, 2)$, $(4, 2)$
Eccentricity: $\frac{1}{2}$

41. Vertices: $(3, 1)$, $(3, 9)$
Minor axis length: 6

42. Foci: $(0, \pm 5)$
Major axis length: 14

43. Center: $(0, 0)$
Major axis: horizontal
Points on the ellipse:
$(3, 1), (4, 0)$

44. Center: $(1, 2)$
Major axis: vertical
Points on the ellipse:
$(1, 6), (3, 2)$

In Exercises 45–52, find the center, foci, and vertices of the hyperbola, and sketch its graph using asymptotes as an aid.

45. $y^2 - \dfrac{x^2}{4} = 1$

46. $\dfrac{x^2}{25} - \dfrac{y^2}{9} = 1$

47. $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{1} = 1$

48. $\dfrac{(y+1)^2}{144} - \dfrac{(x-4)^2}{25} = 1$

49. $9x^2 - y^2 - 36x - 6y + 18 = 0$

50. $y^2 - 9x^2 + 36x - 72 = 0$

51. $x^2 - 9y^2 + 2x - 54y - 80 = 0$

52. $9x^2 - 4y^2 + 54x + 8y + 78 = 0$

In Exercises 53–56, find the center, foci, and vertices of the hyperbola. Use a graphing utility to graph the hyperbola and its asymptotes.

53. $9y^2 - x^2 + 2x + 54y + 62 = 0$

54. $9x^2 - y^2 + 54x + 10y + 55 = 0$

55. $3x^2 - 2y^2 - 6x - 12y - 27 = 0$

56. $3y^2 - x^2 + 6x - 12y = 0$

In Exercises 57–64, find an equation of the hyperbola.

57. Vertices: $(\pm 1, 0)$
Asymptotes: $y = \pm 3x$

58. Vertices: $(0, \pm 3)$
Asymptotes: $y = \pm 3x$

59. Vertices: $(2, \pm 3)$
Point on graph: $(0, 5)$

60. Vertices: $(2, \pm 3)$
Foci: $(2, \pm 5)$

61. Center: $(0, 0)$
Vertex: $(0, 2)$
Focus: $(0, 4)$

62. Center: $(0, 0)$
Vertex: $(3, 0)$
Focus: $(5, 0)$

63. Vertices: $(0, 2), (6, 2)$
Asymptotes: $y = \frac{2}{3}x$
$y = 4 - \frac{2}{3}x$

64. Focus: $(10, 0)$
Asymptotes: $y = \pm\frac{3}{4}x$

In Exercises 65 and 66, find equations for (a) the tangent lines and (b) the normal lines to the hyperbola for the given value of *x*.

65. $\dfrac{x^2}{9} - y^2 = 1, \quad x = 6$

66. $\dfrac{y^2}{4} - \dfrac{x^2}{2} = 1, \quad x = 4$

In Exercises 67–76, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

67. $x^2 + 4y^2 - 6x + 16y + 21 = 0$

68. $4x^2 - y^2 - 4x - 3 = 0$

69. $y^2 - 4y - 4x = 0$

70. $25x^2 - 10x - 200y - 119 = 0$

71. $4x^2 + 4y^2 - 16y + 15 = 0$

72. $y^2 - 4y = x + 5$

73. $9x^2 + 9y^2 - 36x + 6y + 34 = 0$

74. $2x(x - y) = y(3 - y - 2x)$

75. $3(x - 1)^2 = 6 + 2(y + 1)^2$

76. $9(x + 3)^2 = 36 - 4(y - 2)^2$

Writing About Concepts

77. (a) Give the definition of a parabola.
(b) Give the standard forms of a parabola with vertex at (h, k).
(c) In your own words, state the reflective property of a parabola.

78. (a) Give the definition of an ellipse.
(b) Give the standard forms of an ellipse with center at (h, k).

79. (a) Give the definition of a hyperbola.
(b) Give the standard forms of a hyperbola with center at (h, k).
(c) Write equations for the asymptotes of a hyperbola.

80. Define the eccentricity of an ellipse. In your own words, describe how changes in the eccentricity affect the ellipse.

81. ***Solar Collector*** A solar collector for heating water is constructed with a sheet of stainless steel that is formed into the shape of a parabola (see figure). The water will flow through a pipe that is located at the focus of the parabola. At what distance from the vertex is the pipe?

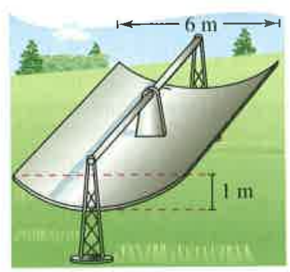


Figure for 81

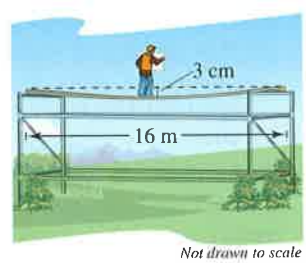


Figure for 82

82. ***Beam Deflection*** A simply supported beam that is 16 meters long has a load concentrated at the center (see figure). The deflection of the beam at its center is 3 centimeters. Assume that the shape of the deflected beam is parabolic.
(a) Find an equation of the parabola. (Assume that the origin is at the center of the beam.)
(b) How far from the center of the beam is the deflection 1 centimeter?

83. Find an equation of the tangent line to the parabola $y = ax^2$ at $x = x_0$. Prove that the x-intercept of this tangent line is $(x_0/2, 0)$.

When sketching (by hand) a curve represented by a set of parametric equations, you can plot points in the xy-plane. Each set of coordinates (x, y) is determined from a value chosen for the parameter t. By plotting the resulting points in order of increasing values of t, the curve is traced out in a specific direction. This is called the **orientation** of the curve.

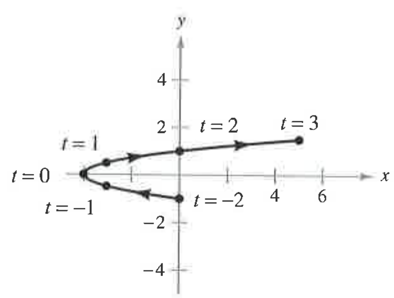
EXAMPLE 1 Sketching a Curve

Sketch the curve described by the parametric equations

$$x = t^2 - 4 \quad \text{and} \quad y = \frac{t}{2}, \quad -2 \le t \le 3.$$

Solution For values of t on the given interval, the parametric equations yield the points (x, y) shown in the table.

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

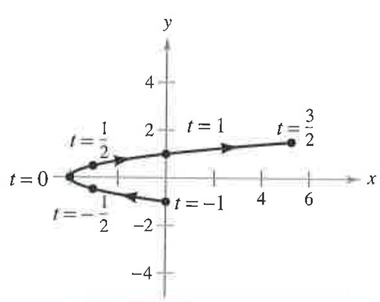


Parametric equations:
$x = t^2 - 4$ and $y = \frac{t}{2}, -2 \le t \le 3$

Figure 10.20

By plotting these points in order of increasing t and using the continuity of f and g, you obtain the curve C shown in Figure 10.20. Note that the arrows on the curve indicate its orientation as t increases from -2 to 3.

NOTE From the Vertical Line Test, you can see that the graph shown in Figure 10.20 does not define y as a function of x. This points out one benefit of parametric equations—they can be used to represent graphs that are more general than graphs of functions.



Parametric equations:
$x = 4t^2 - 4$ and $y = t, -1 \le t \le \frac{3}{2}$

Figure 10.21

It often happens that two different sets of parametric equations have the same graph. For example, the set of parametric equations

$$x = 4t^2 - 4 \quad \text{and} \quad y = t, \quad -1 \le t \le \frac{3}{2}$$

has the same graph as the set given in Example 1. However, comparing the values of t in Figures 10.20 and 10.21, you can see that the second graph is traced out more *rapidly* (considering t as time) than the first graph. So, in applications, different parametric representations can be used to represent various *speeds* at which objects travel along a given path.

TECHNOLOGY Most graphing utilities have a *parametric* graphing mode. If you have access to such a utility, use it to confirm the graphs shown in Figures 10.20 and 10.21. Does the curve given by

$$x = 4t^2 - 8t \quad \text{and} \quad y = 1 - t, \quad -\tfrac{1}{2} \le t \le 2$$

represent the same graph as that shown in Figures 10.20 and 10.21? What do you notice about the *orientation* of this curve?

Eliminating the Parameter

Finding a rectangular equation that represents the graph of a set of parametric equations is called **eliminating the parameter.** For instance, you can eliminate the parameter from the set of parametric equations in Example 1 as follows.

Parametric equations	⇨	Solve for t in one equation.	⇨	Substitute into second equation.	⇨	Rectangular equation
$x = t^2 - 4$ $y = t/2$		$t = 2y$		$x = (2y)^2 - 4$		$x = 4y^2 - 4$

Once you have eliminated the parameter, you can recognize that the equation $x = 4y^2 - 4$ represents a parabola with a horizontal axis and vertex at $(-4, 0)$, as shown in Figure 10.20.

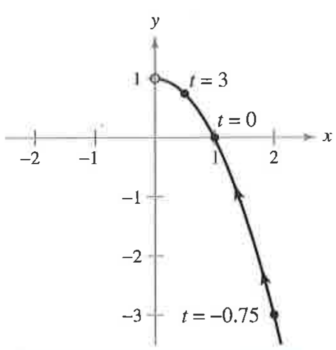
The range of x and y implied by the parametric equations may be altered by the change to rectangular form. In such instances the domain of the rectangular equation must be adjusted so that its graph matches the graph of the parametric equations. Such a situation is demonstrated in the next example.

EXAMPLE 2 Adjusting the Domain After Eliminating the Parameter

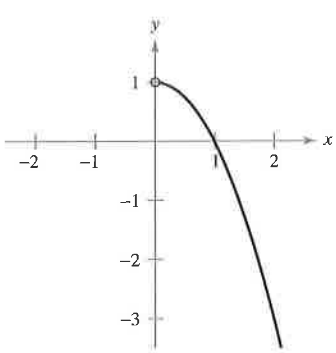
Sketch the curve represented by the equations

$$x = \frac{1}{\sqrt{t+1}} \quad \text{and} \quad y = \frac{t}{t+1}, \quad t > -1$$

by eliminating the parameter and adjusting the domain of the resulting rectangular equation.



Parametric equations:
$x = \frac{1}{\sqrt{t+1}},\ y = \frac{t}{t+1},\ t > -1$



Rectangular equation:
$y = 1 - x^2,\ x > 0$

Figure 10.22

Solution Begin by solving one of the parametric equations for t. For instance, you can solve the first equation for t as follows.

$$x = \frac{1}{\sqrt{t+1}} \qquad \text{Parametric equation for } x$$

$$x^2 = \frac{1}{t+1} \qquad \text{Square each side.}$$

$$t + 1 = \frac{1}{x^2}$$

$$t = \frac{1}{x^2} - 1 = \frac{1 - x^2}{x^2} \qquad \text{Solve for } t.$$

Now, substituting into the parametric equation for y produces

$$y = \frac{t}{t+1} \qquad \text{Parametric equation for } y$$

$$y = \frac{(1 - x^2)/x^2}{[(1 - x^2)/x^2] + 1} \qquad \text{Substitute } (1 - x^2)/x^2 \text{ for } t.$$

$$y = 1 - x^2. \qquad \text{Simplify.}$$

The rectangular equation, $y = 1 - x^2$, is defined for all values of x, but from the parametric equation for x you can see that the curve is defined only when $t > -1$. This implies that you should restrict the domain of x to positive values, as shown in Figure 10.22.

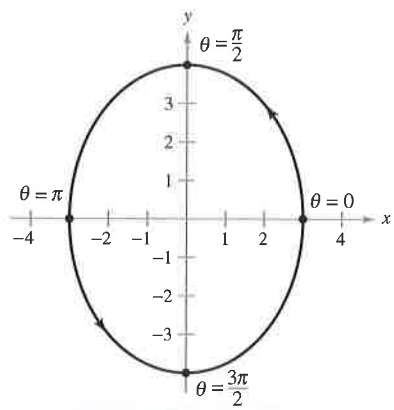
It is not necessary for the parameter in a set of parametric equations to represent time. The next example uses an *angle* as the parameter.

EXAMPLE 3 Using Trigonometry to Eliminate a Parameter

Sketch the curve represented by

$$x = 3\cos\theta \quad \text{and} \quad y = 4\sin\theta, \quad 0 \le \theta \le 2\pi$$

by eliminating the parameter and finding the corresponding rectangular equation.



Parametric equations:
$x = 3\cos\theta,\ y = 4\sin\theta$
Rectangular equation:
$\frac{x^2}{9} + \frac{y^2}{16} = 1$

Figure 10.23

Solution Begin by solving for $\cos\theta$ and $\sin\theta$ in the given equations.

$$\cos\theta = \frac{x}{3} \quad \text{and} \quad \sin\theta = \frac{y}{4}$$ Solve for $\cos\theta$ and $\sin\theta$.

Next, make use of the identity $\sin^2\theta + \cos^2\theta = 1$ to form an equation involving only x and y.

$$\cos^2\theta + \sin^2\theta = 1$$ Trigonometric identity

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$ Substitute.

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$ Rectangular equation

From this rectangular equation you can see that the graph is an ellipse centered at $(0, 0)$, with vertices at $(0, 4)$ and $(0, -4)$ and minor axis of length $2b = 6$, as shown in Figure 10.23. Note that the ellipse is traced out *counterclockwise* as θ varies from 0 to 2π.

Using the technique shown in Example 3, you can conclude that the graph of the parametric equations

$$x = h + a\cos\theta \quad \text{and} \quad y = k + b\sin\theta, \quad 0 \le \theta \le 2\pi$$

is the ellipse (traced counterclockwise) given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

The graph of the parametric equations

$$x = h + a\sin\theta \quad \text{and} \quad y = k + b\cos\theta, \quad 0 \le \theta \le 2\pi$$

is also the ellipse (traced clockwise) given by

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

Use a graphing utility in *parametric* mode to graph several ellipses.

In Examples 2 and 3, it is important to realize that eliminating the parameter is primarily an *aid to curve sketching*. If the parametric equations represent the path of a moving object, the graph alone is not sufficient to describe the object's motion. You still need the parametric equations to tell you the *position*, *direction*, and *speed* at a given time.

Finding Parametric Equations

The first three examples in this section illustrate techniques for sketching the graph represented by a set of parametric equations. You will now investigate the reverse problem. How can you determine a set of parametric equations for a given graph or a given physical description? From the discussion following Example 1, you know that such a representation is not unique. This is demonstrated further in the following example, which finds two different parametric representations for a given graph.

EXAMPLE 4 Finding Parametric Equations for a Given Graph

Find a set of parametric equations to represent the graph of $y = 1 - x^2$, using each of the following parameters.

a. $t = x$ **b.** The slope $m = \dfrac{dy}{dx}$ at the point (x, y)

Solution

a. Letting $x = t$ produces the parametric equations

$$x = t \quad \text{and} \quad y = 1 - x^2 = 1 - t^2.$$

b. To write x and y in terms of the parameter m, you can proceed as follows.

$$m = \frac{dy}{dx} = -2x \qquad \text{Differentiate } y = 1 - x^2.$$

$$x = -\frac{m}{2} \qquad \text{Solve for } x.$$

This produces a parametric equation for x. To obtain a parametric equation for y, substitute $-m/2$ for x in the original equation.

$$y = 1 - x^2 \qquad \text{Write original rectangular equation.}$$

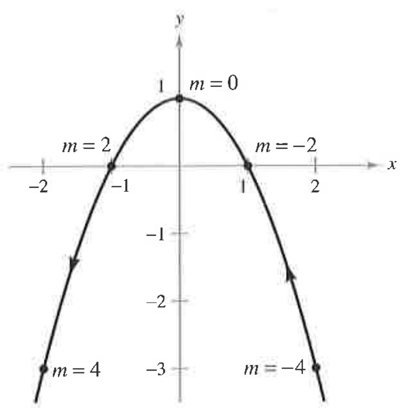
$$y = 1 - \left(-\frac{m}{2}\right)^2 \qquad \text{Substitute } -m/2 \text{ for } x.$$

$$y = 1 - \frac{m^2}{4} \qquad \text{Simplify.}$$

So, the parametric equations are

$$x = -\frac{m}{2} \quad \text{and} \quad y = 1 - \frac{m^2}{4}.$$

In Figure 10.24, note that the resulting curve has a right-to-left orientation as determined by the direction of increasing values of slope m. For part (a), the curve would have the opposite orientation.



Rectangular equation: $y = 1 - x^2$
Parametric equations:
$x = -\dfrac{m}{2}, y = 1 - \dfrac{m^2}{4}$

Figure 10.24

TECHNOLOGY To be efficient at using a graphing utility, it is important that you develop skill in representing a graph by a set of parametric equations. The reason for this is that many graphing utilities have only three graphing modes—(1) functions, (2) parametric equations, and (3) polar equations. Most graphing utilities are not programmed to graph a general equation. For instance, suppose you want to graph the hyperbola $x^2 - y^2 = 1$. To graph the hyperbola in *function* mode, you need two equations: $y = \sqrt{x^2 - 1}$ and $y = -\sqrt{x^2 - 1}$. In *parametric* mode, you can represent the graph by $x = \sec t$ and $y = \tan t$.

CYCLOIDS

Galileo first called attention to the cycloid, once recommending that it be used for the arches of bridges. Pascal once spent 8 days attempting to solve many of the problems of cycloids, such as finding the area under one arch, and the volume of the solid of revolution formed by revolving the curve about a line. The cycloid has so many interesting properties and has caused so many quarrels among mathematicians that it has been called "the Helen of geometry" and "the apple of discord."

FOR FURTHER INFORMATION For more information on cycloids, see the article "The Geometry of Rolling Curves" by John Bloom and Lee Whitt in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

TECHNOLOGY Some graphing utilities allow you to simulate the motion of an object that is moving in the plane or in space. If you have access to such a utility, use it to trace out the path of the cycloid shown in Figure 10.25.

EXAMPLE 5 Parametric Equations for a Cycloid

Determine the curve traced by a point P on the circumference of a circle of radius a rolling along a straight line in a plane. Such a curve is called a **cycloid.**

Solution Let the parameter θ be the measure of the circle's rotation, and let the point $P = (x, y)$ begin at the origin. When $\theta = 0$, P is at the origin. When $\theta = \pi$, P is at a maximum point $(\pi a, 2a)$. When $\theta = 2\pi$, P is back on the x-axis at $(2\pi a, 0)$. From Figure 10.25, you can see that $\angle APC = 180° - \theta$. So,

$$\sin\theta = \sin(180° - \theta) = \sin(\angle APC) = \frac{AC}{a} = \frac{BD}{a}$$

$$\cos\theta = -\cos(180° - \theta) = -\cos(\angle APC) = \frac{AP}{-a}$$

which implies that

$$AP = -a\cos\theta \quad \text{and} \quad BD = a\sin\theta.$$

Because the circle rolls along the x-axis, you know that $OD = \widehat{PD} = a\theta$. Furthermore, because $BA = DC = a$, you have

$$x = OD - BD = a\theta - a\sin\theta$$
$$y = BA + AP = a - a\cos\theta.$$

So, the parametric equations are

$$x = a(\theta - \sin\theta) \quad \text{and} \quad y = a(1 - \cos\theta).$$

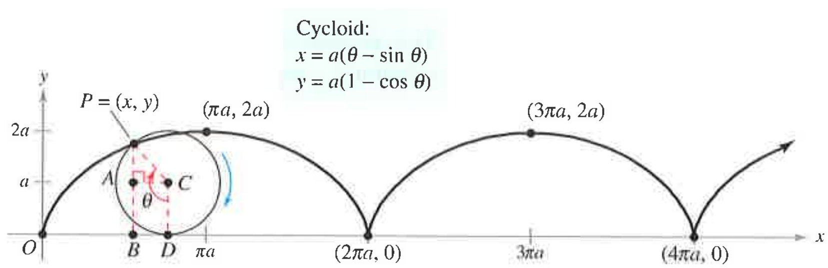


Figure 10.25

The cycloid in Figure 10.25 has sharp corners at the values $x = 2n\pi a$. Notice that the derivatives $x'(\theta)$ and $y'(\theta)$ are both zero at the points for which $\theta = 2n\pi$.

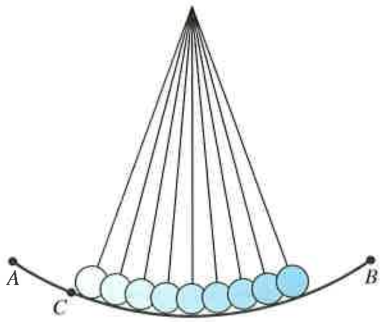
$$x(\theta) = a(\theta - \sin\theta) \qquad y(\theta) = a(1 - \cos\theta)$$
$$x'(\theta) = a - a\cos\theta \qquad y'(\theta) = a\sin\theta$$
$$x'(2n\pi) = 0 \qquad y'(2n\pi) = 0$$

Between these points, the cycloid is called **smooth.**

Definition of a Smooth Curve

A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I is called **smooth** if f' and g' are continuous on I and not simultaneously 0, except possibly at the endpoints of I. The curve C is called **piecewise smooth** if it is smooth on each subinterval of some partition of I.

The Tautochrone and Brachistochrone Problems

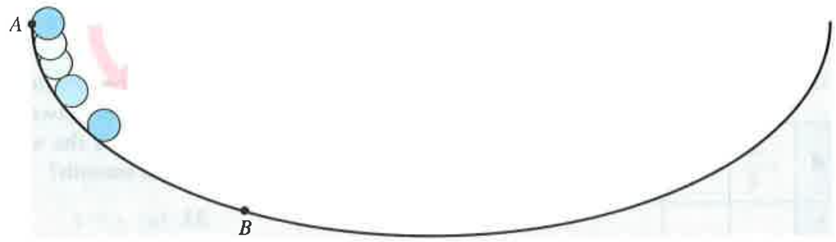


The time required to complete a full swing of the pendulum when starting from point C is only approximately the same as when starting from point A.
Figure 10.26

The type of curve described in Example 5 is related to one of the most famous pairs of problems in the history of calculus. The first problem (called the **tautochrone problem**) began with Galileo's discovery that the time required to complete a full swing of a given pendulum is *approximately* the same whether it makes a large movement at high speed or a small movement at lower speed (see Figure 10.26). Late in his life, Galileo (1564–1642) realized that he could use this principle to construct a clock. However, he was not able to conquer the mechanics of actual construction. Christian Huygens (1629–1695) was the first to design and construct a working model. In his work with pendulums, Huygens realized that a pendulum does not take exactly the same time to complete swings of varying lengths. (This doesn't affect a pendulum clock, because the length of the circular arc is kept constant by giving the pendulum a slight boost each time it passes its lowest point.) But, in studying the problem, Huygens discovered that a ball rolling back and forth on an inverted cycloid does complete each cycle in exactly the same time.

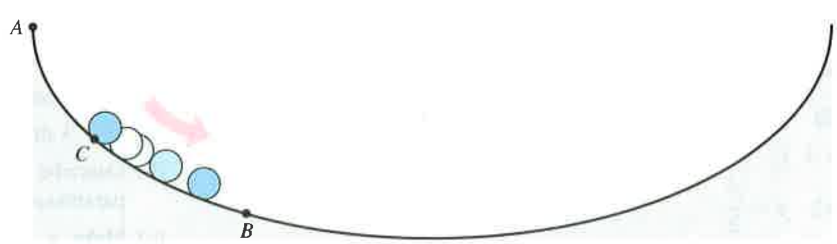
James Bernoulli (1654–1705)

James Bernoulli, also called Jacques, was the older brother of John. He was one of several accomplished mathematicians of the Swiss Bernoulli family. James's mathematical accomplishments have given him a prominent place in the early development of calculus.



An inverted cycloid is the path down which a ball will roll in the shortest time.
Figure 10.27

The second problem, which was posed by John Bernoulli in 1696, is called the **brachistochrone problem**—in Greek, *brachys* means short and chronos means time. The problem was to determine the path down which a particle will slide from point A to point B in the *shortest time*. Several mathematicians took up the challenge, and the following year the problem was solved by Newton, Leibniz, L'Hôpital, John Bernoulli, and James Bernoulli. As it turns out, the solution is not a straight line from A to B, but an inverted cycloid passing through the points A and B, as shown in Figure 10.27. The amazing part of the solution is that a particle starting at rest at *any* other point C of the cycloid between A and B will take exactly the same time to reach B, as shown in Figure 10.28.



A ball starting at point C takes the same time to reach point B as one that starts at point A.
Figure 10.28

FOR FURTHER INFORMATION To see a proof of the famous brachistochrone problem, see the article "A New Minimization Proof for the Brachistochrone" by Gary Lawlor in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

Exercises for Section 10.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. Consider the parametric equations $x = \sqrt{t}$ and $y = 1 - t$.

(a) Complete the table.

t	0	1	2	3	4
x					
y					

(b) Plot the points (x, y) generated in the table, and sketch a graph of the parametric equations. Indicate the orientation of the graph.

(c) Use a graphing utility to confirm your graph in part (b).

(d) Find the rectangular equation by eliminating the parameter, and sketch its graph. Compare the graph in part (b) with the graph of the rectangular equation.

2. Consider the parametric equations $x = 4\cos^2\theta$ and $y = 2\sin\theta$.

(a) Complete the table.

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
x					
y					

(b) Plot the points (x, y) generated in the table, and sketch a graph of the parametric equations. Indicate the orientation of the graph.

(c) Use a graphing utility to confirm your graph in part (b).

(d) Find the rectangular equation by eliminating the parameter, and sketch its graph. Compare the graph in part (b) with the graph of the rectangular equation.

(e) If values of θ were selected from the interval $[\pi/2, 3\pi/2]$ for the table in part (a), would the graph in part (b) be different? Explain.

In Exercises 3–20, sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.

3. $x = 3t - 1, \quad y = 2t + 1$

4. $x = 3 - 2t, \quad y = 2 + 3t$

5. $x = t + 1, \quad y = t^2$

6. $x = 2t^2, \quad y = t^4 + 1$

7. $x = t^3, \quad y = \frac{t^2}{2}$

8. $x = t^2 + t, \quad y = t^2 - t$

9. $x = \sqrt{t}, \quad y = t - 2$

10. $x = \sqrt[4]{t}, \quad y = 3 - t$

11. $x = t - 1, \quad y = \frac{t}{t-1}$

12. $x = 1 + \frac{1}{t}, \quad y = t - 1$

13. $x = 2t, \quad y = |t - 2|$

14. $x = |t - 1|, \quad y = t + 2$

15. $x = e^t, \quad y = e^{3t} + 1$

16. $x = e^{-t}, \quad y = e^{2t} - 1$

17. $x = \sec\theta, \quad y = \cos\theta, \quad 0 \le \theta < \pi/2, \quad \pi/2 < \theta \le \pi$

18. $x = \tan^2\theta, \quad y = \sec^2\theta$

19. $x = 3\cos\theta, \quad y = 3\sin\theta$

20. $x = 2\cos\theta, \quad y = 6\sin\theta$

In Exercises 21–32, use a graphing utility to graph the curve represented by the parametric equations (indicate the orientation of the curve). Eliminate the parameter and write the corresponding rectangular equation.

21. $x = 4\sin 2\theta, y = 2\cos 2\theta$

22. $x = \cos\theta, y = 2\sin 2\theta$

23. $x = 4 + 2\cos\theta$
$y = -1 + \sin\theta$

24. $x = 4 + 2\cos\theta$
$y = -1 + 2\sin\theta$

25. $x = 4 + 2\cos\theta$
$y = -1 + 4\sin\theta$

26. $x = \sec\theta$
$y = \tan\theta$

27. $x = 4\sec\theta, \quad y = 3\tan\theta$

28. $x = \cos^3\theta, \quad y = \sin^3\theta$

29. $x = t^3, \quad y = 3\ln t$

30. $x = \ln 2t, \quad y = t^2$

31. $x = e^{-t}, \quad y = e^{3t}$

32. $x = e^{2t}, \quad y = e^t$

Comparing Plane Curves **In Exercises 33–36, determine any differences between the curves of the parametric equations. Are the graphs the same? Are the orientations the same? Are the curves smooth?**

33. (a) $x = t$
$y = 2t + 1$

(b) $x = \cos\theta$
$y = 2\cos\theta + 1$

(c) $x = e^{-t}$
$y = 2e^{-t} + 1$

(d) $x = e^t$
$y = 2e^t + 1$

34. (a) $x = 2\cos\theta$
$y = 2\sin\theta$

(b) $x = \sqrt{4t^2 - 1}/|t|$
$y = 1/t$

(c) $x = \sqrt{t}$
$y = \sqrt{4 - t}$

(d) $x = -\sqrt{4 - e^{2t}}$
$y = e^t$

35. (a) $x = \cos\theta$
$y = 2\sin^2\theta$
$0 < \theta < \pi$

(b) $x = \cos(-\theta)$
$y = 2\sin^2(-\theta)$
$0 < \theta < \pi$

36. (a) $x = t + 1, y = t^3$

(b) $x = -t + 1, y = (-t)^3$

37. ***Conjecture***

(a) Use a graphing utility to graph the curves represented by the two sets of parametric equations.

$x = 4\cos t \qquad x = 4\cos(-t)$
$y = 3\sin t \qquad y = 3\sin(-t)$

(b) Describe the change in the graph when the sign of the parameter is changed.

(c) Make a conjecture about the change in the graph of parametric equations when the sign of the parameter is changed.

(d) Test your conjecture with another set of parametric equations.

38. ***Writing*** Review Exercises 33–36 and write a short paragraph describing how the graphs of curves represented by different sets of parametric equations can differ even though eliminating the parameter from each yields the same rectangular equation.

In Exercises 39–42, eliminate the parameter and obtain the standard form of the rectangular equation.

39. Line through (x_1, y_1) and (x_2, y_2):
$x = x_1 + t(x_2 - x_1), \quad y = y_1 + t(y_2 - y_1)$

40. Circle: $x = h + r\cos\theta, \quad y = k + r\sin\theta$

41. Ellipse: $x = h + a\cos\theta, \quad y = k + b\sin\theta$

42. Hyperbola: $x = h + a\sec\theta, \quad y = k + b\tan\theta$

In Exercises 43–50, use the results of Exercises 39–42 to find a set of parametric equations for the line or conic.

43. Line: passes through $(0, 0)$ and $(5, -2)$

44. Line: passes through $(1, 4)$ and $(5, -2)$

45. Circle: center: $(2, 1)$; radius: 4

46. Circle: center: $(-3, 1)$; radius: 3

47. Ellipse: vertices: $(\pm 5, 0)$; foci: $(\pm 4, 0)$

48. Ellipse: vertices: $(4, 7), (4, -3)$; foci: $(4, 5), (4, -1)$

49. Hyperbola: vertices: $(\pm 4, 0)$; foci: $(\pm 5, 0)$

50. Hyperbola: vertices: $(0, \pm 1)$; foci: $(0, \pm 2)$

In Exercises 51–54, find two different sets of parametric equations for the rectangular equation.

51. $y = 3x - 2$

52. $y = \dfrac{2}{x - 1}$

53. $y = x^3$

54. $y = x^2$

In Exercises 55–62, use a graphing utility to graph the curve represented by the parametric equations. Indicate the direction of the curve. Identify any points at which the curve is not smooth.

55. Cycloid: $x = 2(\theta - \sin\theta), \quad y = 2(1 - \cos\theta)$

56. Cycloid: $x = \theta + \sin\theta, \quad y = 1 - \cos\theta$

57. Prolate cycloid: $x = \theta - \frac{3}{2}\sin\theta, \quad y = 1 - \frac{3}{2}\cos\theta$

58. Prolate cycloid: $x = 2\theta - 4\sin\theta, \quad y = 2 - 4\cos\theta$

59. Hypocycloid: $x = 3\cos^3\theta, \quad y = 3\sin^3\theta$

60. Curtate cycloid: $x = 2\theta - \sin\theta, \quad y = 2 - \cos\theta$

61. Witch of Agnesi: $x = 2\cot\theta, \quad y = 2\sin^2\theta$

62. Folium of Descartes: $x = \dfrac{3t}{1 + t^3}, \quad y = \dfrac{3t^2}{1 + t^3}$

Writing About Concepts

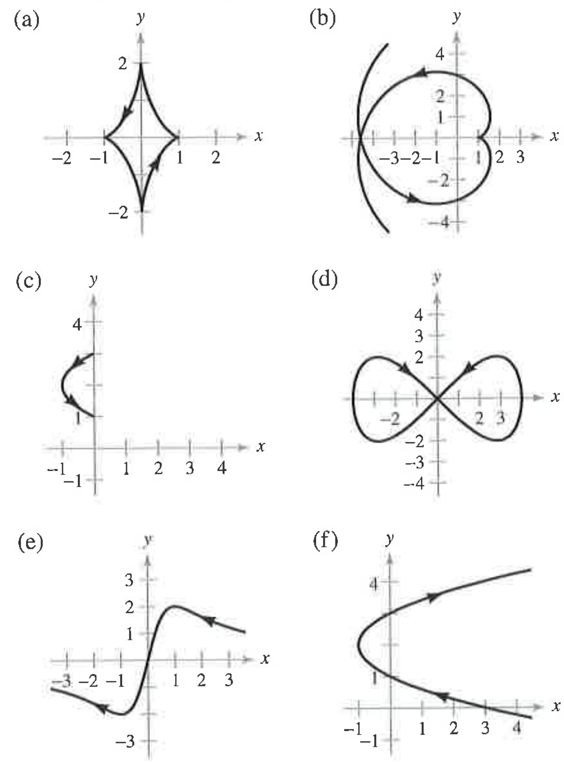
63. State the definition of a plane curve given by parametric equations.

64. Explain the process of sketching a plane curve given by parametric equations. What is meant by the orientation of the curve?

65. State the definition of a smooth curve.

Writing About Concepts *(continued)*

66. Match each set of parametric equations with the correct graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).] Explain your reasoning.



(i) $x = t^2 - 1, \quad y = t + 2$

(ii) $x = \sin^2\theta - 1, \quad y = \sin\theta + 2$

(iii) Lissajous curve: $x = 4\cos\theta, \quad y = 2\sin 2\theta$

(iv) Evolute of ellipse: $x = \cos^3\theta, \quad y = 2\sin^3\theta$

(v) Involute of circle: $x = \cos\theta + \theta\sin\theta$,
$y = \sin\theta - \theta\cos\theta$

(vi) Serpentine curve: $x = \cot\theta, \quad y = 4\sin\theta\cos\theta$

67. ***Curtate Cycloid*** A wheel of radius a rolls along a line without slipping. The curve traced by a point P that is b units from the center $(b < a)$ is called a **curtate cycloid** (see figure). Use the angle θ to find a set of parametric equations for this curve.

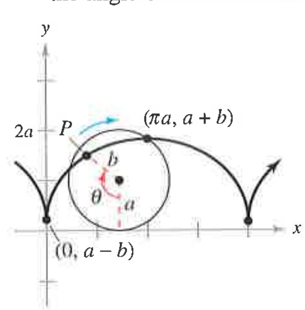


Figure for 67

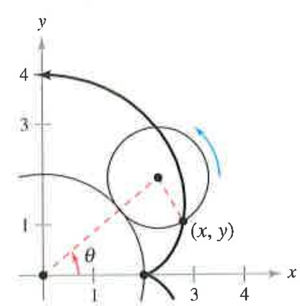


Figure for 68

68. ***Epicycloid*** A circle of radius 1 rolls around the outside of a circle of radius 2 without slipping. The curve traced by a point on the circumference of the smaller circle is called an epicycloid (see figure on previous page). Use the angle θ to find a set of parametric equations for this curve.

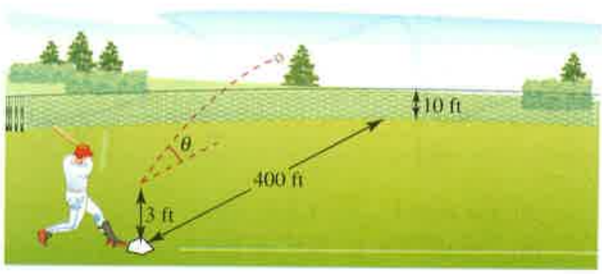
True or False? **In Exercises 69 and 70, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

69. The graph of the parametric equations $x = t^2$ and $y = t^2$ is the line $y = x$.

70. If y is a function of t and x is a function of t, then y is a function of x.

Projectile Motion **In Exercises 71 and 72, consider a projectile launched at a height h feet above the ground and at an angle θ with the horizontal. If the initial velocity is v_0 feet per second, the path of the projectile is modeled by the parametric equations $x = (v_0 \cos \theta)t$ and $y = h + (v_0 \sin \theta)t - 16t^2$.**

71. The center field fence in a ballpark is 10 feet high and 400 feet from home plate. The ball is hit 3 feet above the ground. It leaves the bat at an angle of θ degrees with the horizontal at a speed of 100 miles per hour (see figure).



(a) Write a set of parametric equations for the path of the ball.

(b) Use a graphing utility to graph the path of the ball when $\theta = 15°$. Is the hit a home run?

(c) Use a graphing utility to graph the path of the ball when $\theta = 23°$. Is the hit a home run?

(d) Find the minimum angle at which the ball must leave the bat in order for the hit to be a home run.

72. A rectangular equation for the path of a projectile is $y = 5 + x - 0.005x^2$.

(a) Eliminate the parameter t from the position function for the motion of a projectile to show that the rectangular equation is

$$y = -\frac{16 \sec^2 \theta}{v_0^2} x^2 + (\tan \theta)\, x + h.$$

(b) Use the result of part (a) to find h, v_0, and θ. Find the parametric equations of the path.

(c) Use a graphing utility to graph the rectangular equation for the path of the projectile. Confirm your answer in part (b) by sketching the curve represented by the parametric equations.

(d) Use a graphing utility to approximate the maximum height of the projectile and its range.

Section Project: Cycloids

In Greek, the word *cycloid* means *wheel*, the word *hypocycloid* means *under the wheel*, and the word *epicycloid* means *upon the wheel*. Match the hypocycloid or epicycloid with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

Hypocycloid, H(A, B)

Path traced by a fixed point on a circle of radius B as it rolls around the inside of a circle of radius A

$$x = (A - B)\cos t + B \cos\left(\frac{A - B}{B}\right)t$$

$$y = (A - B)\sin t - B \sin\left(\frac{A - B}{B}\right)t$$

Epicycloid, E(A, B)

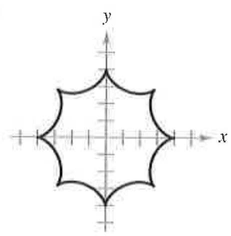
Path traced by a fixed point on a circle of radius B as it rolls around the *outside* of a circle of radius A

$$x = (A + B)\cos t - B \cos\left(\frac{A + B}{B}\right)t$$

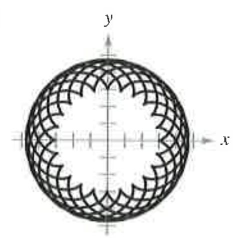
$$y = (A + B)\sin t - B \sin\left(\frac{A + B}{B}\right)t$$

I. $H(8, 3)$	II. $E(8,3)$
III. $H(8, 7)$	IV. $E(24, 3)$
V. $H(24, 7)$	VI. $E(24, 7)$

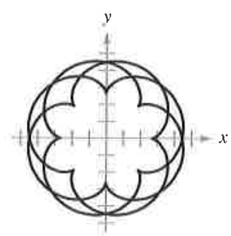
(a)



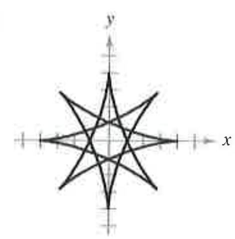
(b)



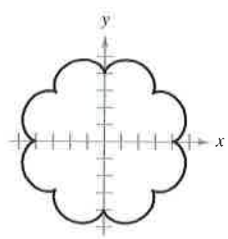
(c)



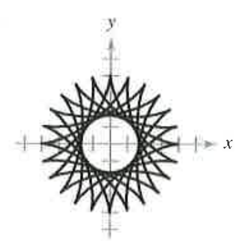
(d)



(e)



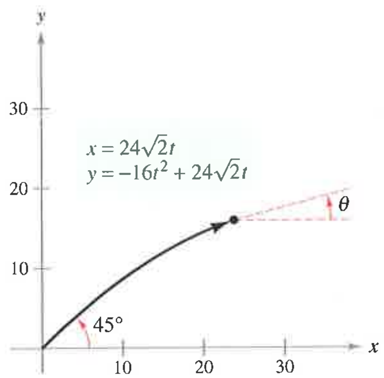
(f)



Exercises based on "Mathematical Discovery via Computer Graphics: Hypocycloids and Epicycloids" by Florence S. Gordon and Sheldon P. Gordon, *College Mathematics Journal*, November 1984, p.441. Used by permission of the authors.

Section 10.3 Parametric Equations and Calculus

- **Find the slope of a tangent line to a curve given by a set of parametric equations.**
- **Find the arc length of a curve given by a set of parametric equations.**
- **Find the area of a surface of revolution (parametric form).**



At time t, the angle of elevation of the projectile is θ, the slope of the tangent line at that point.
Figure 10.29

Slope and Tangent Lines

Now that you can represent a graph in the plane by a set of parametric equations, it is natural to ask how to use calculus to study plane curves. To begin, let's take another look at the projectile represented by the parametric equations

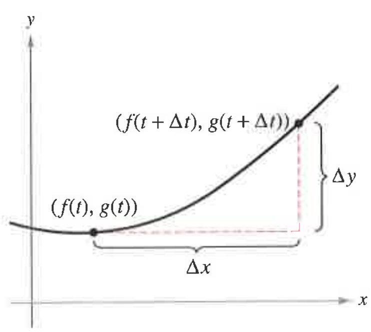
$$x = 24\sqrt{2}t \qquad \text{and} \qquad y = -16t^2 + 24\sqrt{2}t$$

as shown in Figure 10.29. From Section 10.2, you know that these equations enable you to locate the position of the projectile at a given time. You also know that the object is initially projected at an angle of 45°. But how can you find the angle θ representing the object's direction at some other time t? The following theorem answers this question by giving a formula for the slope of the tangent line as a function of t.

THEOREM 10.7 Parametric Form of the Derivative

If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{dx}{dt} \neq 0.$$



The slope of the secant line through the points $(f(t), g(t))$ and $(f(t + \Delta t), g(t + \Delta t))$ is $\Delta y/\Delta x$.
Figure 10.30

Proof In Figure 10.30, consider $\Delta t > 0$ and let

$$\Delta y = g(t + \Delta t) - g(t) \qquad \text{and} \qquad \Delta x = f(t + \Delta t) - f(t).$$

Because $\Delta x \to 0$ as $\Delta t \to 0$, you can write

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}\\ &= \lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{f(t + \Delta t) - f(t)}.\end{aligned}$$

Dividing both the numerator and denominator by Δt, you can use the differentiability of f and g to conclude that

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta t \to 0} \frac{[g(t + \Delta t) - g(t)]/\Delta t}{[f(t + \Delta t) - f(t)]/\Delta t}\\ &= \frac{\displaystyle\lim_{\Delta t \to 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}}{\displaystyle\lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}}\\ &= \frac{g'(t)}{f'(t)}\\ &= \frac{dy/dt}{dx/dt}.\end{aligned}$$

EXAMPLE 1 Differentiation and Parametric Form

Find dy/dx for the curve given by $x = \sin t$ and $y = \cos t$.

STUDY TIP The curve traced out in Example 1 is a circle. Use the formula

$$\frac{dy}{dx} = -\tan t$$

to find the slopes at the points $(1, 0)$ and $(0, 1)$.

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{\cos t} = -\tan t$$

Because dy/dx is a function of t, you can use Theorem 10.7 repeatedly to find *higher-order* derivatives. For instance,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt} \qquad \text{Second derivative}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{dx/dt}. \qquad \text{Third derivative}$$

EXAMPLE 2 Finding Slope and Concavity

For the curve given by

$$x = \sqrt{t} \quad \text{and} \quad y = \frac{1}{4}(t^2 - 4), \quad t \ge 0$$

find the slope and concavity at the point $(2, 3)$.

Solution

Because

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{(1/2)t}{(1/2)t^{-1/2}} = t^{3/2} \qquad \text{Parametric form of first derivative}$$

you can find the second derivative to be

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}[dy/dx]}{dx/dt} = \frac{\frac{d}{dt}[t^{3/2}]}{dx/dt} = \frac{(3/2)t^{1/2}}{(1/2)t^{-1/2}} = 3t. \qquad \text{Parametric form of second derivative}$$

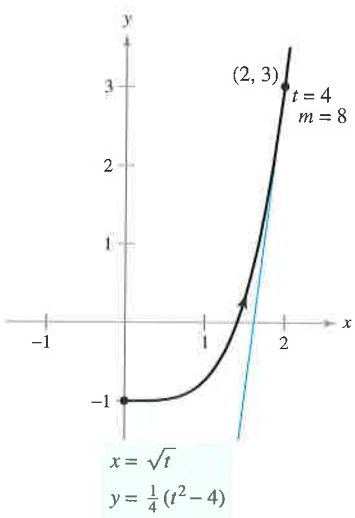
At $(x, y) = (2, 3)$, it follows that $t = 4$, and the slope is

$$\frac{dy}{dx} = (4)^{3/2} = 8.$$

Moreover, when $t = 4$, the second derivative is

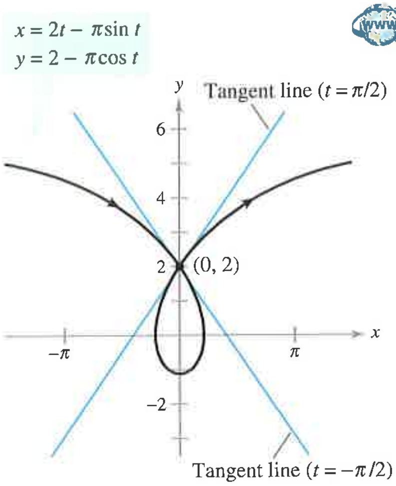
$$\frac{d^2y}{dx^2} = 3(4) = 12 > 0$$

and you can conclude that the graph is concave upward at $(2, 3)$, as shown in Figure 10.31.



The graph is concave upward at $(2, 3)$, when $t = 4$.
Figure 10.31

Because the parametric equations $x = f(t)$ and $y = g(t)$ need not define y as a function of x, it is possible for a plane curve to loop around and cross itself. At such points the curve may have more than one tangent line, as shown in the next example.



This prolate cycloid has two tangent lines at the point (0, 2).
Figure 10.32

EXAMPLE 3 A Curve with Two Tangent Lines at a Point

The **prolate cycloid** given by

$$x = 2t - \pi \sin t \qquad \text{and} \qquad y = 2 - \pi \cos t$$

crosses itself at the point (0, 2), as shown in Figure 10.32. Find the equations of both tangent lines at this point.

Solution Because $x = 0$ and $y = 2$ when $t = \pm\pi/2$, and

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\pi \sin t}{2 - \pi \cos t}$$

you have $dy/dx = -\pi/2$ when $t = -\pi/2$ and $dy/dx = \pi/2$ when $t = \pi/2$. So, the two tangent lines at (0, 2) are

$$y - 2 = -\left(\frac{\pi}{2}\right)x \qquad \text{Tangent line when } t = -\frac{\pi}{2}$$

$$y - 2 = \left(\frac{\pi}{2}\right)x. \qquad \text{Tangent line when } t = \frac{\pi}{2}$$

If $dy/dt = 0$ and $dx/dt \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a horizontal tangent at $(f(t_0), g(t_0))$. For instance, in Example 3, the given curve has a horizontal tangent at the point $(0, 2 - \pi)$ (when $t = 0$). Similarly, if $dx/dt = 0$ and $dy/dt \neq 0$ when $t = t_0$, the curve represented by $x = f(t)$ and $y = g(t)$ has a vertical tangent at $(f(t_0), g(t_0))$.

Arc Length

You have seen how parametric equations can be used to describe the path of a particle moving in the plane. You will now develop a formula for determining the *distance* traveled by the particle along its path.

Recall from Section 7.4 that the formula for the arc length of a curve C given by $y = h(x)$ over the interval $[x_0, x_1]$ is

$$\begin{aligned} s &= \int_{x_0}^{x_1} \sqrt{1 + [h'(x)]^2}\, dx \\ &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \end{aligned}$$

If C is represented by the parametric equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, and if $dx/dt = f'(t) > 0$, you can write

$$\begin{aligned} s = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx &= \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, dx \\ &= \int_a^b \sqrt{\frac{(dx/dt)^2 + (dy/dt)^2}{(dx/dt)^2}}\, \frac{dx}{dt}\, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt. \end{aligned}$$

NOTE When applying the arc length formula to a curve, be sure that the curve is traced out only once on the interval of integration. For instance, the circle given by $x = \cos t$ and $y = \sin t$ is traced out once on the interval $0 \le t \le 2\pi$, but is traced out twice on the interval $0 \le t \le 4\pi$.

THEOREM 10.8 Arc Length in Parametric Form

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \le t \le b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

In the preceding section you saw that if a circle rolls along a line, a point on its circumference will trace a path called a cycloid. If the circle rolls around the circumference of another circle, the path of the point is an **epicycloid.** The next example shows how to find the arc length of an epicycloid.

Arch of a Cycloid

The arc length of an arch of a cycloid was first calculated in 1658 by British architect and mathematician Christopher Wren, famous for rebuilding many buildings and churches in London, including St. Paul's Cathedral.

EXAMPLE 4 Finding Arc Length

A circle of radius 1 rolls around the circumference of a larger circle of radius 4, as shown in Figure 10.33. The epicycloid traced by a point on the circumference of the smaller circle is given by

$$x = 5\cos t - \cos 5t$$

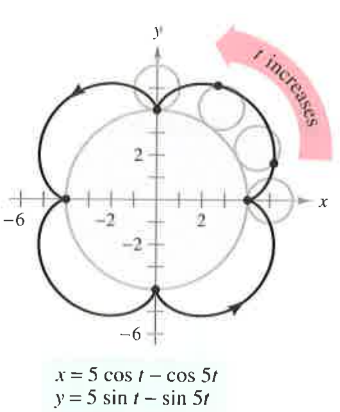
and

$$y = 5\sin t - \sin 5t.$$

Find the distance traveled by the point in one complete trip about the larger circle.

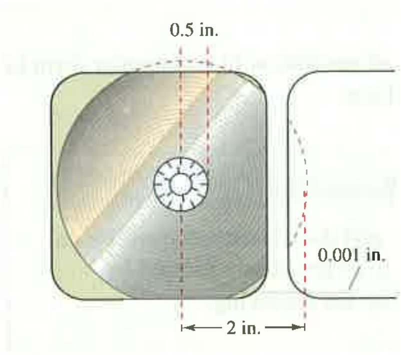
Solution Before applying Theorem 10.8, note in Figure 10.33 that the curve has sharp points when $t = 0$ and $t = \pi/2$. Between these two points, dx/dt and dy/dt are not simultaneously 0. So, the portion of the curve generated from $t = 0$ to $t = \pi/2$ is smooth. To find the total distance traveled by the point, you can find the arc length of that portion lying in the first quadrant and multiply by 4.

$$\begin{aligned}
s &= 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt && \text{Parametric form for arc length}\\
&= 4\int_0^{\pi/2} \sqrt{(-5\sin t + 5\sin 5t)^2 + (5\cos t - 5\cos 5t)^2}\, dt\\
&= 20\int_0^{\pi/2} \sqrt{2 - 2\sin t\sin 5t - 2\cos t\cos 5t}\, dt\\
&= 20\int_0^{\pi/2} \sqrt{2 - 2\cos 4t}\, dt\\
&= 20\int_0^{\pi/2} \sqrt{4\sin^2 2t}\, dt && \text{Trigonometric identity}\\
&= 40\int_0^{\pi/2} \sin 2t\, dt\\
&= -20\Big[\cos 2t\Big]_0^{\pi/2}\\
&= 40
\end{aligned}$$



An epicycloid is traced by a point on the smaller circle as it rolls around the larger circle.
Figure 10.33

For the epicycloid shown in Figure 10.33, an arc length of 40 seems about right because the circumference of a circle of radius 6 is $2\pi r = 12\pi \approx 37.7$.



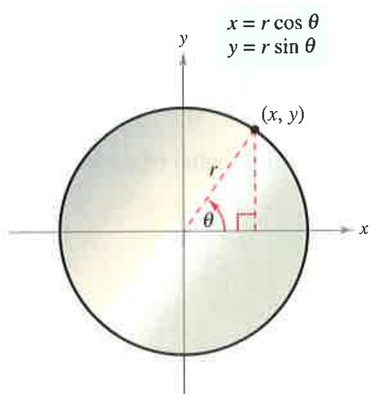


Figure 10.34

EXAMPLE 5 Length of a Recording Tape

A recording tape 0.001 inch thick is wound around a reel whose inner radius is 0.5 inch and whose outer radius is 2 inches, as shown in Figure 10.34. How much tape is required to fill the reel?

Solution To create a model for this problem, assume that as the tape is wound around the reel its distance r from the center increases linearly at a rate of 0.001 inch per revolution, or

$$r = (0.001)\frac{\theta}{2\pi} = \frac{\theta}{2000\pi}, \qquad 1000\pi \le \theta \le 4000\pi$$

where θ is measured in radians. You can determine the coordinates of the point (x, y) corresponding to a given radius to be

$$x = r\cos\theta$$

and

$$y = r\sin\theta.$$

Substituting for r, you obtain the parametric equations

$$x = \left(\frac{\theta}{2000\pi}\right)\cos\theta \quad \text{and} \quad y = \left(\frac{\theta}{2000\pi}\right)\sin\theta.$$

You can use the arc length formula to determine the total length of the tape to be

$$\begin{aligned}
s &= \int_{1000\pi}^{4000\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \\
&= \frac{1}{2000\pi}\int_{1000\pi}^{4000\pi} \sqrt{(-\theta\sin\theta + \cos\theta)^2 + (\theta\cos\theta + \sin\theta)^2}\, d\theta \\
&= \frac{1}{2000\pi}\int_{1000\pi}^{4000\pi} \sqrt{\theta^2 + 1}\, d\theta \\
&= \frac{1}{2000\pi}\left(\frac{1}{2}\right)\left[\theta\sqrt{\theta^2+1} + \ln\left|\theta + \sqrt{\theta^2+1}\right|\right]_{1000\pi}^{4000\pi} \\
&\approx 11{,}781 \text{ inches} \\
&\approx 982 \text{ feet}
\end{aligned}$$

Integration tables (Appendix B), Formula 26

FOR FURTHER INFORMATION For more information on the mathematics of recording tape, see "Tape Counters" by Richard L. Roth in *The American Mathematical Monthly*. To view this article, go to the website *www.matharticles.com*.

NOTE The graph of $r = a\theta$ is called the **spiral of Archimedes**. The graph of $r = \theta/2000\pi$ (in Example 5) is of this form.

The length of the tape in Example 5 can be approximated by adding the circumferences of circular pieces of tape. The smallest circle has a radius of 0.501 and the largest has a radius of 2.

$$\begin{aligned}
s &\approx 2\pi(0.501) + 2\pi(0.502) + 2\pi(0.503) + \cdots + 2\pi(2.000) \\
&= \sum_{i=1}^{1500} 2\pi(0.5 + 0.001i) \\
&= 2\pi[1500(0.5) + 0.001(1500)(1501)/2] \\
&\approx 11{,}786 \text{ inches}
\end{aligned}$$

Area of a Surface of Revolution

You can use the formula for the area of a surface of revolution in rectangular form to develop a formula for surface area in parametric form.

THEOREM 10.9 Area of a Surface of Revolution

If a smooth curve C given by $x = f(t)$ and $y = g(t)$ does not cross itself on an interval $a \le t \le b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following.

1. $S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ Revolution about the x-axis: $g(t) \ge 0$

2. $S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ Revolution about the y-axis: $f(t) \ge 0$

These formulas are easy to remember if you think of the differential of arc length as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Then the formulas are written as follows.

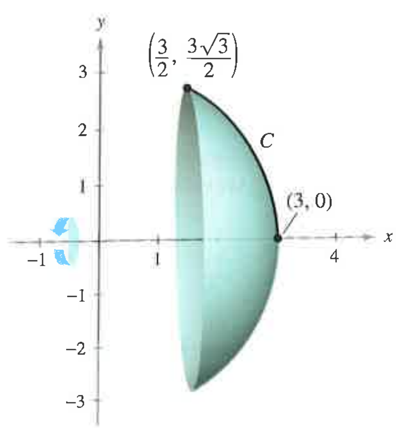
1. $S = 2\pi \int_a^b g(t)\, ds$ **2.** $S = 2\pi \int_a^b f(t)\, ds$

EXAMPLE 6 Finding the Area of a Surface of Revolution

Let C be the arc of the circle

$$x^2 + y^2 = 9$$

from $(3, 0)$ to $(3/2, 3\sqrt{3}/2)$, as shown in Figure 10.35. Find the area of the surface formed by revolving C about the x-axis.



This surface of revolution has a surface area of 9π.
Figure 10.35

Solution You can represent C parametrically by the equations

$$x = 3\cos t \quad \text{and} \quad y = 3\sin t, \quad 0 \le t \le \pi/3.$$

(Note that you can determine the interval for t by observing that $t = 0$ when $x = 3$ and $t = \pi/3$ when $x = 3/2$.) On this interval, C is smooth and y is nonnegative, and you can apply Theorem 10.9 to obtain a surface area of

$$\begin{aligned} S &= 2\pi \int_0^{\pi/3} (3\sin t)\sqrt{(-3\sin t)^2 + (3\cos t)^2}\, dt && \text{Formula for area of a surface of revolution} \\ &= 6\pi \int_0^{\pi/3} \sin t \sqrt{9(\sin^2 t + \cos^2 t)}\, dt \\ &= 6\pi \int_0^{\pi/3} 3\sin t\, dt && \text{Trigonometric identity} \\ &= -18\pi\Big[\cos t\Big]_0^{\pi/3} \\ &= -18\pi\left(\frac{1}{2} - 1\right) \\ &= 9\pi. \end{aligned}$$

Exercises for Section 10.3

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, find dy/dx.

1. $x = t^2,\ y = 5 - 4t$

2. $x = \sqrt[3]{t},\ y = 4 - t$

3. $x = \sin^2\theta,\ y = \cos^2\theta$

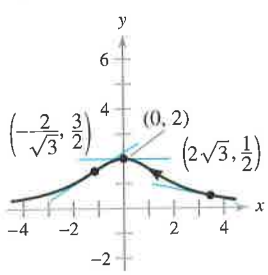
4. $x = 2e^{\theta},\ y = e^{-\theta/2}$

In Exercises 5–14, find dy/dx and d^2y/dx^2, and find the slope and concavity (if possible) at the given value of the parameter.

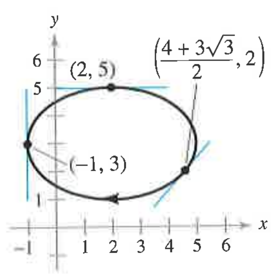
Parametric Equations	*Point*
5. $x = 2t,\ y = 3t - 1$	$t = 3$
6. $x = \sqrt{t},\ y = 3t - 1$	$t = 1$
7. $x = t + 1,\ y = t^2 + 3t$	$t = -1$
8. $x = t^2 + 3t + 2,\ y = 2t$	$t = 0$
9. $x = 2\cos\theta,\ y = 2\sin\theta$	$\theta = \dfrac{\pi}{4}$
10. $x = \cos\theta,\ y = 3\sin\theta$	$\theta = 0$
11. $x = 2 + \sec\theta,\ y = 1 + 2\tan\theta$	$\theta = \dfrac{\pi}{6}$
12. $x = \sqrt{t},\ y = \sqrt{t - 1}$	$t = 2$
13. $x = \cos^3\theta,\ y = \sin^3\theta$	$\theta = \dfrac{\pi}{4}$
14. $x = \theta - \sin\theta,\ y = 1 - \cos\theta$	$\theta = \pi$

In Exercises 15 and 16, find an equation of the tangent line at each given point on the curve.

15. $x = 2\cot\theta$
$y = 2\sin^2\theta$



16. $x = 2 - 3\cos\theta$
$y = 3 + 2\sin\theta$



In Exercises 17–20, (a) use a graphing utility to graph the curve represented by the parametric equations, (b) use a graphing utility to find dx/dt, dy/dt, and dy/dx at the given value of the parameter, (c) find an equation of the tangent line to the curve at the given value of the parameter, and (d) use a graphing utility to graph the curve and the tangent line from part (c).

Parametric Equations	*Parameter*
17. $x = 2t,\ y = t^2 - 1$	$t = 2$
18. $x = t - 1,\ y = \dfrac{1}{t} + 1$	$t = 1$
19. $x = t^2 - t + 2,\ y = t^3 - 3t$	$t = -1$
20. $x = 4\cos\theta,\ y = 3\sin\theta$	$\theta = \dfrac{3\pi}{4}$

In Exercises 21–24, find the equations of the tangent lines at the point where the curve crosses itself.

21. $x = 2\sin 2t,\quad y = 3\sin t$

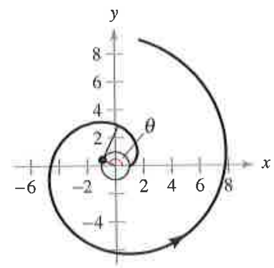
22. $x = 2 - \pi\cos t,\quad y = 2t - \pi\sin t$

23. $x = t^2 - t,\quad y = t^3 - 3t - 1$

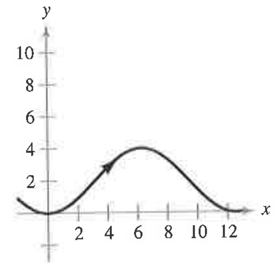
24. $x = t^3 - 6t,\quad y = t^2$

In Exercises 25 and 26, find all points (if any) of horizontal and vertical tangency to the portion of the curve shown.

25. Involute of a circle:
$x = \cos\theta + \theta\sin\theta$
$y = \sin\theta - \theta\cos\theta$



26. $x = 2\theta$
$y = 2(1 - \cos\theta)$



In Exercises 27–36, find all points (if any) of horizontal and vertical tangency to the curve. Use a graphing utility to confirm your results.

27. $x = 1 - t,\quad y = t^2$

28. $x = t + 1,\quad y = t^2 + 3t$

29. $x = 1 - t,\quad y = t^3 - 3t$

30. $x = t^2 - t + 2,\quad y = t^3 - 3t$

31. $x = 3\cos\theta,\quad y = 3\sin\theta$

32. $x = \cos\theta,\quad y = 2\sin 2\theta$

33. $x = 4 + 2\cos\theta,\quad y = -1 + \sin\theta$

34. $x = 4\cos^2\theta,\quad y = 2\sin\theta$

35. $x = \sec\theta,\quad y = \tan\theta$

36. $x = \cos^2\theta,\quad y = \cos\theta$

In Exercises 37–42, determine the t intervals on which the curve is concave downward or concave upward.

37. $x = t^2,\quad y = t^3 - t$

38. $x = 2 + t^2,\quad y = t^2 + t^3$

39. $x = 2t + \ln t,\quad y = 2t - \ln t$

40. $x = t^2,\quad y = \ln t$

41. $x = \sin t,\quad y = \cos t,\quad 0 < t < \pi$

42. $x = 2\cos t,\quad y = \sin t,\quad 0 < t < 2\pi$

Arc Length **In Exercises 43–46, write an integral that represents the arc length of the curve on the given interval. Do not evaluate the integral.**

Parametric Equations	*Interval*
43. $x = 2t - t^2, \quad y = 2t^{3/2}$	$1 \le t \le 2$
44. $x = \ln t, \quad y = t + 1$	$1 \le t \le 6$
45. $x = e^t + 2, \quad y = 2t + 1$	$-2 \le t \le 2$
46. $x = t + \sin t, \quad y = t - \cos t$	$0 \le t \le \pi$

Arc Length **In Exercises 47–52, find the arc length of the curve on the given interval.**

Parametric Equations	*Interval*
47. $x = t^2, \quad y = 2t$	$0 \le t \le 2$
48. $x = t^2 + 1, \quad y = 4t^3 + 3$	$-1 \le t \le 0$
49. $x = e^{-t} \cos t, \quad y = e^{-t} \sin t$	$0 \le t \le \dfrac{\pi}{2}$
50. $x = \arcsin t, \quad y = \ln\sqrt{1 - t^2}$	$0 \le t \le \frac{1}{2}$
51. $x = \sqrt{t}, \quad y = 3t - 1$	$0 \le t \le 1$
52. $x = t, \quad y = \dfrac{t^5}{10} + \dfrac{1}{6t^3}$	$1 \le t \le 2$

Arc Length **In Exercises 53–56, find the arc length of the curve on the interval $[0, 2\pi]$.**

53. Hypocycloid perimeter: $x = a \cos^3 \theta, y = a \sin^3 \theta$

54. Circle circumference: $x = a \cos \theta, y = a \sin \theta$

55. Cycloid arch: $x = a(\theta - \sin \theta), y = a(1 - \cos \theta)$

56. Involute of a circle: $x = \cos \theta + \theta \sin \theta, y = \sin \theta - \theta \cos \theta$

57. ***Path of a Projectile*** The path of a projectile is modeled by the parametric equations

$$x = (90 \cos 30°)t \quad \text{and} \quad y = (90 \sin 30°)t - 16t^2$$

where x and y are measured in feet.

(a) Use a graphing utility to graph the path of the projectile.

(b) Use a graphing utility to approximate the range of the projectile.

(c) Use the integration capabilities of a graphing utility to approximate the arc length of the path. Compare this result with the range of the projectile.

58. ***Path of a Projectile*** If the projectile in Exercise 57 is launched at an angle θ with the horizontal, its parametric equations are

$$x = (90 \cos \theta)t \quad \text{and} \quad y = (90 \sin \theta)t - 16t^2.$$

Use a graphing utility to find the angle that maximizes the range of the projectile. What angle maximizes the arc length of the trajectory?

59. ***Folium of Descartes*** Consider the parametric equations

$$x = \frac{4t}{1 + t^3} \quad \text{and} \quad y = \frac{4t^2}{1 + t^3}.$$

(a) Use a graphing utility to graph the curve represented by the parametric equations.

(b) Use a graphing utility to find the points of horizontal tangency to the curve.

(c) Use the integration capabilities of a graphing utility to approximate the arc length of the closed loop. (*Hint:* Use symmetry and integrate over the interval $0 \le t \le 1$.)

60. ***Witch of Agnesi*** Consider the parametric equations

$$x = 4 \cot \theta \quad \text{and} \quad y = 4 \sin^2 \theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

(a) Use a graphing utility to graph the curve represented by the parametric equations.

(b) Use a graphing utility to find the points of horizontal tangency to the curve.

(c) Use the integration capabilities of a graphing utility to approximate the arc length over the interval $\pi/4 \le \theta \le \pi/2$.

61. ***Writing***

(a) Use a graphing utility to graph each set of parametric equations.

$x = t - \sin t$	$x = 2t - \sin(2t)$
$y = 1 - \cos t$	$y = 1 - \cos(2t)$
$0 \le t \le 2\pi$	$0 \le t \le \pi$

(b) Compare the graphs of the two sets of parametric equations in part (a). If the curve represents the motion of a particle and t is time, what can you infer about the average speeds of the particle on the paths represented by the two sets of parametric equations?

(c) Without graphing the curve, determine the time required for a particle to traverse the same path as in parts (a) and (b) if the path is modeled by

$$x = \tfrac{1}{2}t - \sin\left(\tfrac{1}{2}t\right) \quad \text{and} \quad y = 1 - \cos\left(\tfrac{1}{2}t\right).$$

62. ***Writing***

(a) Each set of parametric equations represents the motion of a particle. Use a graphing utility to graph each set.

First Particle	*Second Particle*
$x = 3 \cos t$	$x = 4 \sin t$
$y = 4 \sin t$	$y = 3 \cos t$
$0 \le t \le 2\pi$	$0 \le t \le 2\pi$

(b) Determine the number of points of intersection.

(c) Will the particles ever be at the same place at the same time? If so, identify the points.

(d) Explain what happens if the motion of the second particle is represented by

$$x = 2 + 3 \sin t, \quad y = 2 - 4 \cos t, \quad 0 \le t \le 2\pi.$$

Surface Area **In Exercises 63–66, write an integral that represents the area of the surface generated by revolving the curve about the x-axis. Use a graphing utility to approximate the integral.**

Parametric Equations	*Interval*
63. $x = 4t, \quad y = t + 1$	$0 \le t \le 2$
64. $x = \frac{1}{4}t^2, \quad y = t + 2$	$0 \le t \le 4$
65. $x = \cos^2 \theta, \quad y = \cos \theta$	$0 \le \theta \le \frac{\pi}{2}$
66. $x = \theta + \sin \theta, \quad y = \theta + \cos \theta$	$0 \le \theta \le \frac{\pi}{2}$

Surface Area **In Exercises 67–72, find the area of the surface generated by revolving the curve about each given axis.**

67. $x = t, y = 2t, \quad 0 \le t \le 4,$ (a) x-axis (b) y-axis

68. $x = t, y = 4 - 2t, \quad 0 \le t \le 2,$ (a) x-axis (b) y-axis

69. $x = 4 \cos \theta, y = 4 \sin \theta, \quad 0 \le \theta \le \frac{\pi}{2}, \quad$ y-axis

70. $x = \frac{1}{3}t^3, y = t + 1, \quad 1 \le t \le 2, \quad$ y-axis

71. $x = a \cos^3 \theta, y = a \sin^3 \theta, \quad 0 \le \theta \le \pi, \quad$ x-axis

72. $x = a \cos \theta, y = b \sin \theta, \quad 0 \le \theta \le 2\pi,$

(a) x-axis (b) y-axis

Writing About Concepts

73. Give the parametric form of the derivative.

74. Mentally determine dy/dx.

(a) $x = t, \quad y = 4$ (b) $x = t, \quad y = 4t - 3$

75. Sketch a graph of a curve defined by the parametric equations $x = g(t)$ and $y = f(t)$ such that $dx/dt > 0$ and $dy/dt < 0$ for all real numbers t.

76. Sketch a graph of a curve defined by the parametric equations $x = g(t)$ and $y = f(t)$ such that $dx/dt < 0$ and $dy/dt < 0$ for all real numbers t.

77. Give the integral formula for arc length in parametric form.

78. Give the integral formulas for the areas of the surfaces of revolution formed when a smooth curve C is revolved about (a) the x-axis and (b) the y-axis.

79. Use integration by substitution to show that if y is a continuous function of x on the interval $a \le x \le b$, where $x = f(t)$ and $y = g(t)$, then

$$\int_a^b y\,dx = \int_{t_1}^{t_2} g(t)f'(t)\,dt$$

where $f(t_1) = a, f(t_2) = b$, and both g and f' are continuous on $[t_1, t_2]$.

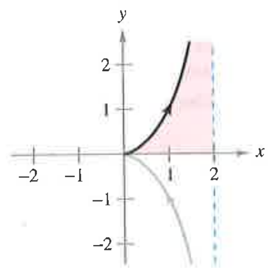
80. ***Surface Area*** A portion of a sphere of radius r is removed by cutting out a circular cone with its vertex at the center of the sphere. The vertex of the cone forms an angle of 2θ. Find the surface area removed from the sphere.

Area **In Exercises 81 and 82, find the area of the region. (Use the result of Exercise 79.)**

81. $x = 2 \sin^2 \theta$

$y = 2 \sin^2 \theta \tan \theta$

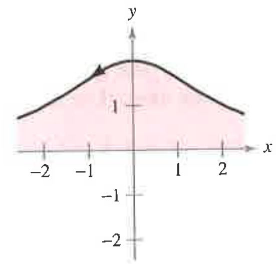
$0 \le \theta < \frac{\pi}{2}$



82. $x = 2 \cot \theta$

$y = 2 \sin^2 \theta$

$0 < \theta < \pi$



Areas of Simple Closed Curves **In Exercises 83–88, use a computer algebra system and the result of Exercise 79 to match the closed curve with its area. (These exercises were adapted from the article "The Surveyor's Area Formula" by Bart Braden in the September 1986 issue of the *College Mathematics Journal*, by permission of the author.)**

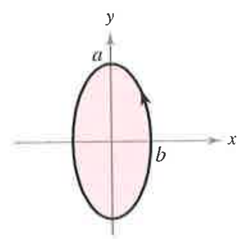
(a) $\frac{8}{3}ab$ **(b)** $\frac{3}{8}\pi a^2$ **(c)** $2\pi a^2$

(d) πab **(e)** $2\pi ab$ **(f)** $6\pi a^2$

83. Ellipse: $(0 \le t \le 2\pi)$

$x = b \cos t$

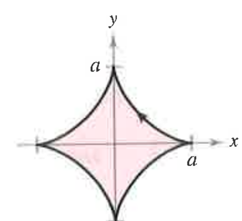
$y = a \sin t$



84. Astroid: $(0 \le t \le 2\pi)$

$x = a \cos^3 t$

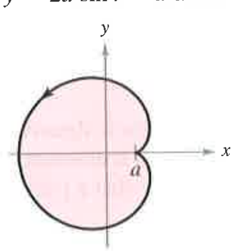
$y = a \sin^3 t$



85. Cardioid: $(0 \le t \le 2\pi)$

$x = 2a \cos t - a \cos 2t$

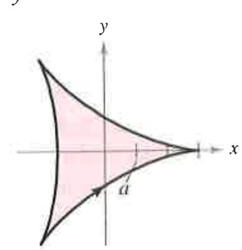
$y = 2a \sin t - a \sin 2t$



86. Deltoid: $(0 \le t \le 2\pi)$

$x = 2a \cos t + a \cos 2t$

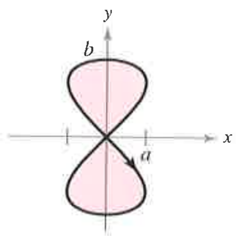
$y = 2a \sin t - a \sin 2t$



87. Hourglass: $(0 \le t \le 2\pi)$

$x = a \sin 2t$

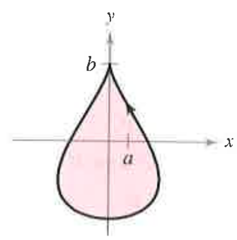
$y = b \sin t$



88. Teardrop: $(0 \le t \le 2\pi)$

$x = 2a \cos t - a \sin 2t$

$y = b \sin t$



Centroid **In Exercises 89 and 90, find the centroid of the region bounded by the graph of the parametric equations and the coordinate axes. (Use the result of Exercise 79.)**

89. $x = \sqrt{t},\ y = 4 - t$

90. $x = \sqrt{4 - t},\ y = \sqrt{t}$

Volume **In Exercises 91 and 92, find the volume of the solid formed by revolving the region bounded by the graphs of the given equations about the x-axis. (Use the result of Exercise 79.)**

91. $x = 3 \cos \theta, \quad y = 3 \sin \theta$

92. $x = \cos \theta, \quad y = 3 \sin \theta, \quad a > 0$

93. ***Cycloid*** Use the parametric equations

$$x = a(\theta - \sin \theta) \quad \text{and} \quad y = a(1 - \cos \theta),\ a > 0$$

to answer the following.

(a) Find dy/dx and d^2y/dx^2.

(b) Find the equations of the tangent line at the point where $\theta = \pi/6$.

(c) Find all points (if any) of horizontal tangency.

(d) Determine where the curve is concave upward or concave downward.

(e) Find the length of one arc of the curve.

94. Use the parametric equations

$$x = t^2\sqrt{3} \quad \text{and} \quad y = 3t - \frac{1}{3}t^3$$

to answer the following.

(a) Use a graphing utility to graph the curve on the interval $-3 \le t \le 3$.

(b) Find dy/dx and d^2y/dx^2.

(c) Find the equation of the tangent line at the point $\left(\sqrt{3}, \frac{8}{3}\right)$.

(d) Find the length of the curve.

(e) Find the surface area generated by revolving the curve about the x-axis.

95. ***Involute of a Circle*** The involute of a circle is described by the endpoint P of a string that is held taut as it is unwound from a spool that does not turn (see figure). Show that a parametric representation of the involute is

$$x = r(\cos \theta + \theta \sin \theta) \quad \text{and} \quad y = r(\sin \theta - \theta \cos \theta).$$

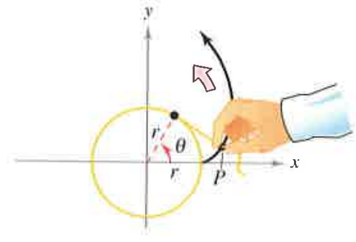


Figure for 95

96. ***Involute of a Circle*** The figure shows a piece of string tied to a circle with a radius of one unit. The string is just long enough to reach the opposite side of the circle. Find the area that is covered when the string is unwound counterclockwise.

97. (a) Use a graphing utility to graph the curve given by

$$x = \frac{1 - t^2}{1 + t^2}, \quad y = \frac{2t}{1 + t^2}, \qquad -20 \le t \le 20.$$

(b) Describe the graph and confirm your result analytically.

(c) Discuss the speed at which the curve is traced as t increases from -20 to 20.

98. ***Tractrix*** A person moves from the origin along the positive y-axis pulling a weight at the end of a 12-meter rope. Initially, the weight is located at the point $(12, 0)$.

(a) In Exercise 86 of Section 8.7, it was shown that the path of the weight is modeled by the rectangular equation

$$y = -12 \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$$

where $0 < x \le 12$. Use a graphing utility to graph the rectangular equation.

(b) Use a graphing utility to graph the parametric equations

$$x = 12 \operatorname{sech} \frac{t}{12} \quad \text{and} \quad y = t - 12 \tanh \frac{t}{12}$$

where $t \ge 0$. How does this graph compare with the graph in part (a)? Which graph (if either) do you think is a better representation of the path?

(c) Use the parametric equations for the tractrix to verify that the distance from the y-intercept of the tangent line to the point of tangency is independent of the location of the point of tangency.

True or False? **In Exercises 99 and 100, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

99. If $x = f(t)$ and $y = g(t)$, then $d^2y/dx^2 = g''(t)/f''(t)$.

100. The curve given by $x = t^3$, $y = t^2$ has a horizontal tangent at the origin because $dy/dt = 0$ when $t = 0$.

Section 10.4

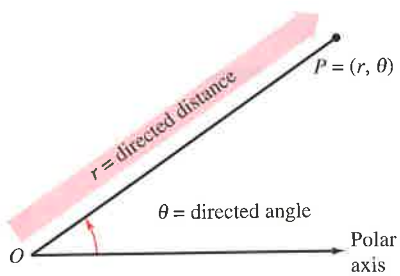
Polar Coordinates and Polar Graphs

- **Understand the polar coordinate system.**
- **Rewrite rectangular coordinates and equations in polar form and vice versa.**
- **Sketch the graph of an equation given in polar form.**
- **Find the slope of a tangent line to a polar graph.**
- **Identify several types of special polar graphs.**

Polar Coordinates

So far, you have been representing graphs as collections of points (x, y) on the rectangular coordinate system. The corresponding equations for these graphs have been in either rectangular or parametric form. In this section you will study a coordinate system called the **polar coordinate system.**

To form the polar coordinate system in the plane, fix a point O, called the **pole** (or **origin**), and construct from O an initial ray called the **polar axis,** as shown in Figure 10.36. Then each point P in the plane can be assigned **polar coordinates** (r, θ), as follows.



Polar coordinates
Figure 10.36

r = *directed distance* from O to P

θ = *directed angle*, counterclockwise from polar axis to segment $\overline{OP}$

Figure 10.37 shows three points on the polar coordinate system. Notice that in this system, it is convenient to locate points with respect to a grid of concentric circles intersected by **radial lines** through the pole.

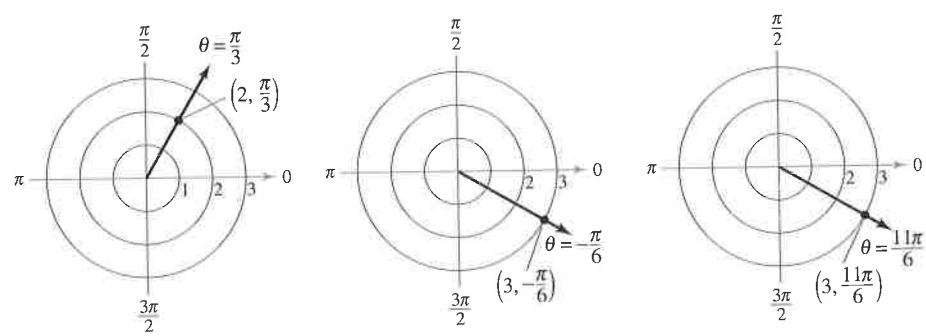


Figure 10.37

With rectangular coordinates, each point (x, y) has a unique representation. This is not true with polar coordinates. For instance, the coordinates (r, θ) and $(r, 2\pi + \theta)$ represent the same point [see parts (b) and (c) in Figure 10.37]. Also, because r is a *directed distance*, the coordinates (r, θ) and $(-r, \theta + \pi)$ represent the same point. In general, the point (r, θ) can be written as

$$(r, \theta) = (r, \theta + 2n\pi)$$

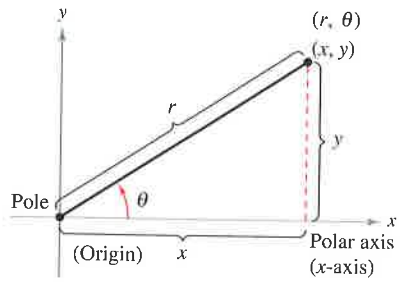
or

$$(r, \theta) = (-r, \theta + (2n + 1)\pi)$$

where n is any integer. Moreover, the pole is represented by $(0, \theta)$, where θ is any angle.

Polar Coordinates

The mathematician credited with first using polar coordinates was James Bernoulli, who introduced them in 1691. However, there is some evidence that it may have been Isaac Newton who first used them.



Relating polar and rectangular coordinates
Figure 10.38

Coordinate Conversion

To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive x-axis and the pole with the origin, as shown in Figure 10.38. Because (x, y) lies on a circle of radius r, it follows that $r^2 = x^2 + y^2$. Moreover, for $r > 0$, the definition of the trigonometric functions implies that

$$\tan\theta = \frac{y}{x}, \qquad \cos\theta = \frac{x}{r}, \qquad \text{and} \qquad \sin\theta = \frac{y}{r}.$$

If $r < 0$, you can show that the same relationships hold.

THEOREM 10.10 Coordinate Conversion

The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.

1. $x = r\cos\theta$ **2.** $\tan\theta = \dfrac{y}{x}$

$y = r\sin\theta$ $r^2 = x^2 + y^2$

EXAMPLE 1 Polar-to-Rectangular Conversion

a. For the point $(r, \theta) = (2, \pi)$,

$$x = r\cos\theta = 2\cos\pi = -2 \qquad \text{and} \qquad y = r\sin\theta = 2\sin\pi = 0.$$

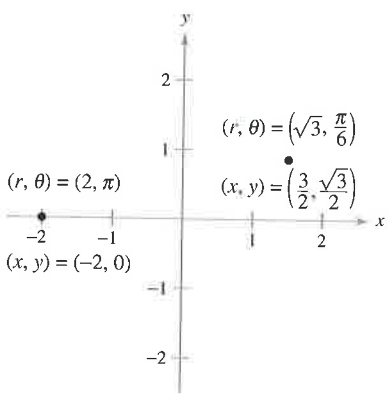
So, the rectangular coordinates are $(x, y) = (-2, 0)$.

b. For the point $(r, \theta) = (\sqrt{3}, \pi/6)$,

$$x = \sqrt{3}\cos\frac{\pi}{6} = \frac{3}{2} \qquad \text{and} \qquad y = \sqrt{3}\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

So, the rectangular coordinates are $(x, y) = (3/2, \sqrt{3}/2)$.

See Figure 10.39.



To convert from polar to rectangular coordinates, let $x = r\cos\theta$ and $y = r\sin\theta$.
Figure 10.39

EXAMPLE 2 Rectangular-to-Polar Conversion

a. For the second quadrant point $(x, y) = (-1, 1)$,

$$\tan\theta = \frac{y}{x} = -1 \quad \Longrightarrow \quad \theta = \frac{3\pi}{4}.$$

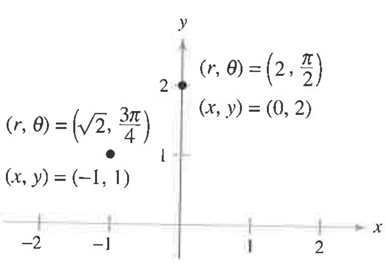
Because θ was chosen to be in the same quadrant as (x, y), you should use a positive value of r.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-1)^2 + (1)^2} \\ &= \sqrt{2} \end{aligned}$$

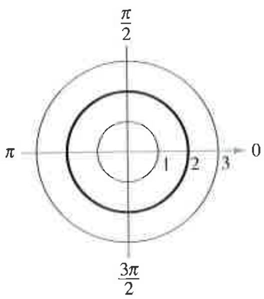
This implies that *one* set of polar coordinates is $(r, \theta) = (\sqrt{2}, 3\pi/4)$.

b. Because the point $(x, y) = (0, 2)$ lies on the positive y-axis, choose $\theta = \pi/2$ and $r = 2$, and one set of polar coordinates is $(r, \theta) = (2, \pi/2)$.

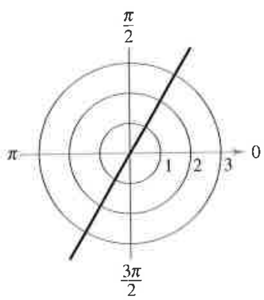
See Figure 10.40.



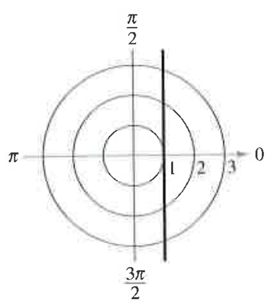
To convert from rectangular to polar coordinates, let $\tan\theta = y/x$ and $r = \sqrt{x^2 + y^2}$.
Figure 10.40



(a) Circle: $r = 2$



(b) Radial line: $\theta = \dfrac{\pi}{3}$



(c) Vertical line: $r = \sec\theta$
Figure 10.41

Polar Graphs

One way to sketch the graph of a polar equation is to convert to rectangular coordinates and then sketch the graph of the rectangular equation.

EXAMPLE 3 Graphing Polar Equations

Describe the graph of each polar equation. Confirm each description by converting to a rectangular equation.

a. $r = 2$ **b.** $\theta = \dfrac{\pi}{3}$ **c.** $r = \sec\theta$

Solution

a. The graph of the polar equation $r = 2$ consists of all points that are two units from the pole. In other words, this graph is a circle centered at the origin with a radius of 2. [See Figure 10.41(a).] You can confirm this by using the relationship $r^2 = x^2 + y^2$ to obtain the rectangular equation

$$x^2 + y^2 = 2^2. \qquad \text{Rectangular equation}$$

b. The graph of the polar equation $\theta = \pi/3$ consists of all points on the line that makes an angle of $\pi/3$ with the positive x-axis. [See Figure 10.41(b).] You can confirm this by using the relationship $\tan\theta = y/x$ to obtain the rectangular equation

$$y = \sqrt{3}\,x. \qquad \text{Rectangular equation}$$

c. The graph of the polar equation $r = \sec\theta$ is not evident by simple inspection, so you can begin by converting to rectangular form using the relationship $r\cos\theta = x$.

$$r = \sec\theta \qquad \text{Polar equation}$$
$$r\cos\theta = 1$$
$$x = 1 \qquad \text{Rectangular equation}$$

From the rectangular equation, you can see that the graph is a vertical line. [See Figure 10.41(c).]

TECHNOLOGY Sketching the graphs of complicated polar equations *by hand* can be tedious. With technology, however, the task is not difficult. If your graphing utility has a *polar* mode, use it to graph the equations in the exercise set. If your graphing utility doesn't have a *polar* mode, but does have a *parametric* mode, you can graph $r = f(\theta)$ by writing the equation as

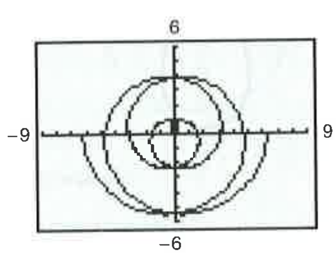
$$x = f(\theta)\cos\theta$$
$$y = f(\theta)\sin\theta.$$

For instance, the graph of $r = \frac{1}{2}\theta$ shown in Figure 10.42 was produced with a graphing calculator in *parametric* mode. This equation was graphed using the parametric equations

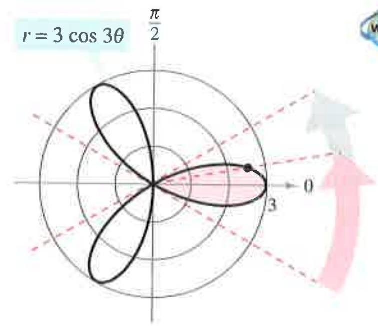
$$x = \frac{1}{2}\theta\cos\theta$$

$$y = \frac{1}{2}\theta\sin\theta$$

with the values of θ varying from -4π to 4π. This curve is of the form $r = a\theta$ and is called a **spiral of Archimedes.**



Spiral of Archimedes
Figure 10.42



The area of one petal of the rose curve that lies between the radial lines $\theta = -\pi/6$ and $\theta = \pi/6$ is $3\pi/4$.
Figure 10.51

EXAMPLE 1 Finding the Area of a Polar Region

Find the area of one petal of the rose curve given by $r = 3\cos 3\theta$.

Solution In Figure 10.51, you can see that the right petal is traced as θ increases from $-\pi/6$ to $\pi/6$. So, the area is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta = \frac{1}{2}\int_{-\pi/6}^{\pi/6} (3\cos 3\theta)^2\,d\theta$$

Formula for area in polar coordinates

$$= \frac{9}{2}\int_{-\pi/6}^{\pi/6} \frac{1+\cos 6\theta}{2}\,d\theta$$

Trigonometric identity

$$= \frac{9}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_{-\pi/6}^{\pi/6}$$

$$= \frac{9}{4}\left(\frac{\pi}{6} + \frac{\pi}{6}\right)$$

$$= \frac{3\pi}{4}.$$

NOTE To find the area of the region lying inside all three petals of the rose curve in Example 1, you could not simply integrate between 0 and 2π. In doing this you would obtain $9\pi/2$, which is twice the area of the three petals. The duplication occurs because the rose curve is traced twice as θ increases from 0 to 2π.

EXAMPLE 2 Finding the Area Bounded by a Single Curve

Find the area of the region lying between the inner and outer loops of the limaçon $r = 1 - 2\sin\theta$.

Solution In Figure 10.52, note that the inner loop is traced as θ increases from $\pi/6$ to $5\pi/6$. So, the area inside the *inner loop* is

$$A_1 = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 - 2\sin\theta)^2\,d\theta$$

Formula for area in polar coordinates

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 - 4\sin\theta + 4\sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[1 - 4\sin\theta + 4\left(\frac{1-\cos 2\theta}{2}\right)\right]d\theta$$

Trigonometric identity

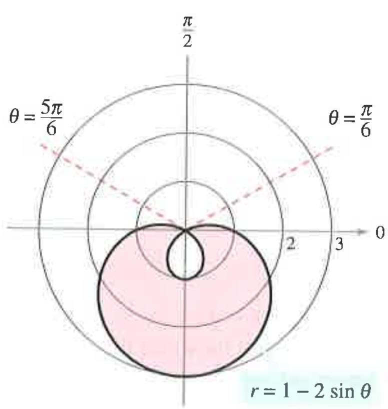
$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3 - 4\sin\theta - 2\cos 2\theta)\,d\theta$$

Simplify.

$$= \frac{1}{2}\left[3\theta + 4\cos\theta - \sin 2\theta\right]_{\pi/6}^{5\pi/6}$$

$$= \frac{1}{2}\left(2\pi - 3\sqrt{3}\right)$$

$$= \pi - \frac{3\sqrt{3}}{2}.$$



The area between the inner and outer loops is approximately 8.34.
Figure 10.52

In a similar way, you can integrate from $5\pi/6$ to $13\pi/6$ to find that the area of the region lying inside the outer loop is $A_2 = 2\pi + (3\sqrt{3}/2)$. The area of the region lying between the two loops is the difference of A_2 and A_1.

$$A = A_2 - A_1 = \left(2\pi + \frac{3\sqrt{3}}{2}\right) - \left(\pi - \frac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3} \approx 8.34$$

Points of Intersection of Polar Graphs

Because a point may be represented in different ways in polar coordinates, care must be taken in determining the points of intersection of two polar graphs. For example, consider the points of intersection of the graphs of

$$r = 1 - 2\cos\theta \quad \text{and} \quad r = 1$$

as shown in Figure 10.53. If, as with rectangular equations, you attempted to find the points of intersection by solving the two equations simultaneously, you would obtain

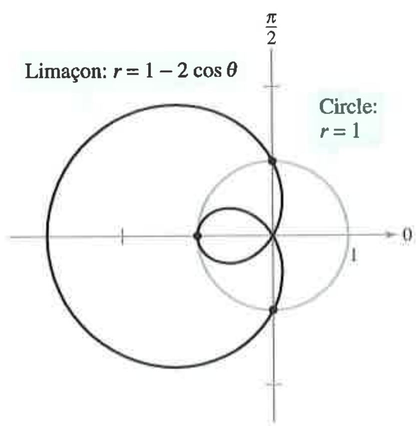
$$\begin{aligned} r &= 1 - 2\cos\theta && \text{First equation} \\ 1 &= 1 - 2\cos\theta && \text{Substitute } r = 1 \text{ from 2nd equation into 1st equation.} \\ \cos\theta &= 0 && \text{Simplify.} \\ \theta &= \frac{\pi}{2}, \frac{3\pi}{2}. && \text{Solve for } \theta. \end{aligned}$$

FOR FURTHER INFORMATION For more information on using technology to find points of intersection, see the article "Finding Points of Intersection of Polar-Coordinate Graphs" by Warren W. Esty in *Mathematics Teacher*. To view this article, go to the website *www.matharticles.com*.

The corresponding points of intersection are $(1, \pi/2)$ and $(1, 3\pi/2)$. However, from Figure 10.53 you can see that there is a *third* point of intersection that did not show up when the two polar equations were solved simultaneously. (This is one reason why you should sketch a graph when finding the area of a polar region.) The reason the third point was not found is that it does not occur with the same coordinates in the two graphs. On the graph of $r = 1$, the point occurs with coordinates $(1, \pi)$, but on the graph of $r = 1 - 2\cos\theta$, the point occurs with coordinates $(-1, 0)$.

You can compare the problem of finding points of intersection of two polar graphs with that of finding collision points of two satellites in intersecting orbits about Earth, as shown in Figure 10.54. The satellites will not collide as long as they reach the points of intersection at different times (θ-values). Collisions will occur only at the points of intersection that are "simultaneous points"—those reached at the same time (θ-value).

NOTE Because the pole can be represented by $(0, \theta)$, where θ is *any* angle, you should check separately for the pole when finding points of intersection.



Three points of intersection: $(1, \pi/2)$, $(-1, 0)$, $(1, 3\pi/2)$
Figure 10.53

The paths of satellites can cross without causing a collision.
Figure 10.54

EXAMPLE 3 Finding the Area of a Region Between Two Curves

Find the area of the region common to the two regions bounded by the following curves.

$r = -6\cos\theta$ Circle

$r = 2 - 2\cos\theta$ Cardioid

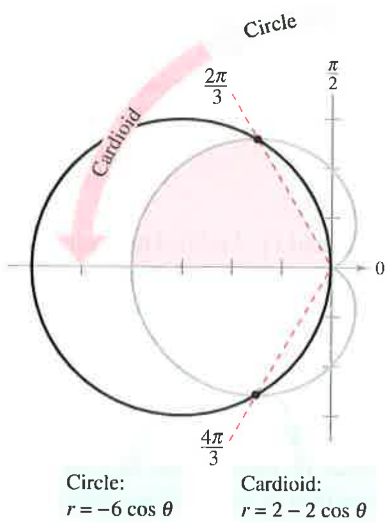


Figure 10.55

Solution Because both curves are symmetric with respect to the x-axis, you can work with the upper half-plane, as shown in Figure 10.55. The gray shaded region lies between the circle and the radial line $\theta = 2\pi/3$. Because the circle has coordinates $(0, \pi/2)$ at the pole, you can integrate between $\pi/2$ and $2\pi/3$ to obtain the area of this region. The region that is shaded red is bounded by the radial lines $\theta = 2\pi/3$ and $\theta = \pi$ and the cardioid. So, you can find the area of this second region by integrating between $2\pi/3$ and π. The sum of these two integrals gives the area of the common region lying *above* the radial line $\theta = \pi$.

$$\frac{A}{2} = \underbrace{\frac{1}{2}\int_{\pi/2}^{2\pi/3} (-6\cos\theta)^2\, d\theta}_{\text{Region between circle and radial line } \theta = 2\pi/3} + \underbrace{\frac{1}{2}\int_{2\pi/3}^{\pi} (2 - 2\cos\theta)^2\, d\theta}_{\text{Region between cardioid and radial lines } \theta = 2\pi/3 \text{ and } \theta = \pi}$$

$$= 18\int_{\pi/2}^{2\pi/3} \cos^2\theta\, d\theta + \frac{1}{2}\int_{2\pi/3}^{\pi} (4 - 8\cos\theta + 4\cos^2\theta)\, d\theta$$

$$= 9\int_{\pi/2}^{2\pi/3} (1 + \cos 2\theta)\, d\theta + \int_{2\pi/3}^{\pi} (3 - 4\cos\theta + \cos 2\theta)\, d\theta$$

$$= 9\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/2}^{2\pi/3} + \left[3\theta - 4\sin\theta + \frac{\sin 2\theta}{2}\right]_{2\pi/3}^{\pi}$$

$$= 9\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{2}\right) + \left(3\pi - 2\pi + 2\sqrt{3} + \frac{\sqrt{3}}{4}\right)$$

$$= \frac{5\pi}{2}$$

$$\approx 7.85$$

Finally, multiplying by 2, you can conclude that the total area is 5π.

NOTE To check the reasonableness of the result obtained in Example 3, note that the area of the circular region is $\pi r^2 = 9\pi$. So, it seems reasonable that the area of the region lying inside the circle and the cardioid is 5π.

To see the benefit of polar coordinates for finding the area in Example 3, consider the following integral, which gives the comparable area in rectangular coordinates.

$$\frac{A}{2} = \int_{-4}^{-3/2} \sqrt{2\sqrt{1 - 2x} - x^2 - 2x + 2}\, dx + \int_{-3/2}^{0} \sqrt{-x^2 - 6x}\, dx$$

Use the integration capabilities of a graphing utility to show that you obtain the same area as that found in Example 3.

Arc Length in Polar Form

NOTE When applying the arc length formula to a polar curve, be sure that the curve is traced out only once on the interval of integration. For instance, the rose curve given by $r = \cos 3\theta$ is traced out once on the interval $0 \le \theta \le \pi$, but is traced out twice on the interval $0 \le \theta \le 2\pi$.

The formula for the length of a polar arc can be obtained from the arc length formula for a curve described by parametric equations. (See Exercise 77.)

THEOREM 10.14 Arc Length of a Polar Curve

Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The length of the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta.$$

EXAMPLE 4 Finding the Length of a Polar Curve

Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the cardioid

$$r = f(\theta) = 2 - 2\cos\theta$$

as shown in Figure 10.56.

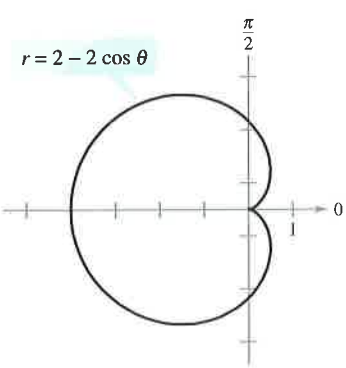


Figure 10.56

Solution Because $f'(\theta) = 2\sin\theta$, you can find the arc length as follows.

$$\begin{aligned}
s &= \int_\alpha^\beta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta && \text{Formula for arc length of a polar curve}\\
&= \int_0^{2\pi} \sqrt{(2 - 2\cos\theta)^2 + (2\sin\theta)^2}\,d\theta\\
&= 2\sqrt{2}\int_0^{2\pi} \sqrt{1 - \cos\theta}\,d\theta && \text{Simplify.}\\
&= 2\sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\frac{\theta}{2}}\,d\theta && \text{Trigonometric identity}\\
&= 4\int_0^{2\pi} \sin\frac{\theta}{2}\,d\theta && \sin\frac{\theta}{2} \ge 0 \text{ for } 0 \le \theta \le 2\pi\\
&= 8\left[-\cos\frac{\theta}{2}\right]_0^{2\pi}\\
&= 8(1 + 1)\\
&= 16
\end{aligned}$$

In the fifth step of the solution, it is legitimate to write

$$\sqrt{2\sin^2(\theta/2)} = \sqrt{2}\sin(\theta/2)$$

rather than

$$\sqrt{2\sin^2(\theta/2)} = \sqrt{2}\,|\sin(\theta/2)|$$

because $\sin(\theta/2) \ge 0$ for $0 \le \theta \le 2\pi$.

NOTE Using Figure 10.56, you can determine the reasonableness of this answer by comparing it with the circumference of a circle. For example, a circle of radius $\frac{5}{2}$ has a circumference of $5\pi \approx 15.7$.

Area of a Surface of Revolution

The polar coordinate versions of the formulas for the area of a surface of revolution can be obtained from the parametric versions given in Theorem 10.9, using the equations $x = r\cos\theta$ and $y = r\sin\theta$.

NOTE When using Theorem 10.15, check to see that the graph of $r = f(\theta)$ is traced only once on the interval $\alpha \le \theta \le \beta$. For example, the circle given by $r = \cos\theta$ is traced only once on the interval $0 \le \theta \le \pi$.

THEOREM 10.15 Area of a Surface of Revolution

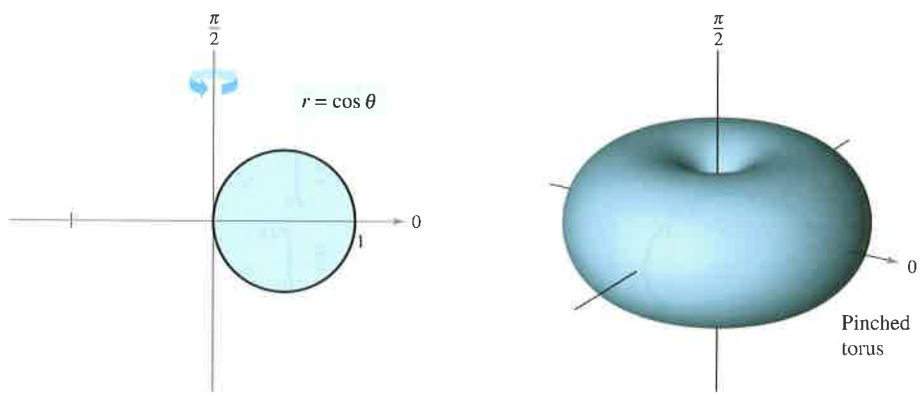
Let f be a function whose derivative is continuous on an interval $\alpha \le \theta \le \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line is as follows.

1. $S = 2\pi \displaystyle\int_\alpha^\beta f(\theta)\sin\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$ About the polar axis

2. $S = 2\pi \displaystyle\int_\alpha^\beta f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta$ About the line $\theta = \frac{\pi}{2}$

EXAMPLE 5 Finding the Area of a Surface of Revolution

Find the area of the surface formed by revolving the circle $r = f(\theta) = \cos\theta$ about the line $\theta = \pi/2$, as shown in Figure 10.57.



(a) (b)

Figure 10.57

Solution You can use the second formula given in Theorem 10.15 with $f'(\theta) = -\sin\theta$. Because the circle is traced once as θ increases from 0 to π, you have

$$S = 2\pi \int_\alpha^\beta f(\theta)\cos\theta\sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\,d\theta \qquad \text{Formula for area of a surface of revolution}$$

$$= 2\pi \int_0^\pi \cos\theta(\cos\theta)\sqrt{\cos^2\theta + \sin^2\theta}\,d\theta$$

$$= 2\pi \int_0^\pi \cos^2\theta\,d\theta \qquad \text{Trigonometric identity}$$

$$= \pi \int_0^\pi (1 + \cos 2\theta)\,d\theta \qquad \text{Trigonometric identity}$$

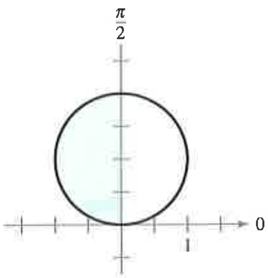
$$= \pi\left[\theta + \frac{\sin 2\theta}{2}\right]_0^\pi = \pi^2.$$

Exercises for Section 10.5

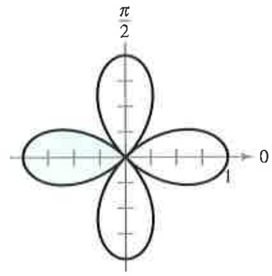
See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, write an integral that represents the area of the shaded region shown in the figure. Do not evaluate the integral.

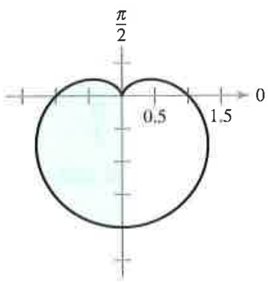
1. $r = 2 \sin \theta$



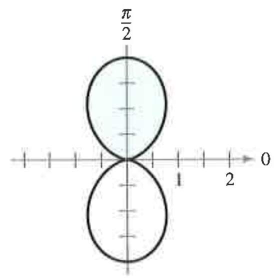
2. $r = \cos 2\theta$



3. $r = 1 - \sin \theta$



4. $r = 1 - \cos 2\theta$



In Exercises 5 and 6, find the area of the region bounded by the graph of the polar equation using (a) a geometric formula and (b) integration.

5. $r = 8 \sin \theta$

6. $r = 3 \cos \theta$

In Exercises 7–12, find the area of the region.

7. One petal of $r = 2 \cos 3\theta$

8. One petal of $r = 6 \sin 2\theta$

9. One petal of $r = \cos 2\theta$

10. One petal of $r = \cos 5\theta$

11. Interior of $r = 1 - \sin \theta$

12. Interior of $r = 1 - \sin \theta$ (above the polar axis)

In Exercises 13–16, use a graphing utility to graph the polar equation and find the area of the given region.

13. Inner loop of $r = 1 + 2 \cos \theta$

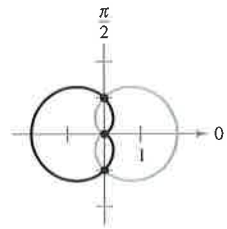
14. Inner loop of $r = 4 - 6 \sin \theta$

15. Between the loops of $r = 1 + 2 \cos \theta$

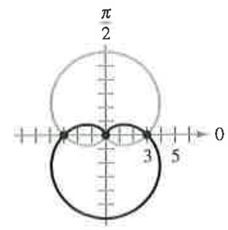
16. Between the loops of $r = 2(1 + 2 \sin \theta)$

In Exercises 17–26, find the points of intersection of the graphs of the equations.

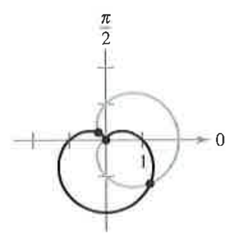
17. $r = 1 + \cos \theta$
$r = 1 - \cos \theta$



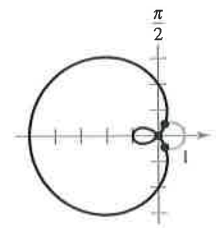
18. $r = 3(1 + \sin \theta)$
$r = 3(1 - \sin \theta)$



19. $r = 1 + \cos \theta$
$r = 1 - \sin \theta$



20. $r = 2 - 3 \cos \theta$
$r = \cos \theta$



21. $r = 4 - 5 \sin \theta$
$r = 3 \sin \theta$

22. $r = 1 + \cos \theta$
$r = 3 \cos \theta$

23. $r = \dfrac{\theta}{2}$
$r = 2$

24. $\theta = \dfrac{\pi}{4}$
$r = 2$

25. $r = 4 \sin 2\theta$
$r = 2$

26. $r = 3 + \sin \theta$
$r = 2 \csc \theta$

In Exercises 27 and 28, use a graphing utility to approximate the points of intersection of the graphs of the polar equations. Confirm your results analytically.

27. $r = 2 + 3 \cos \theta$
$r = \dfrac{\sec \theta}{2}$

28. $r = 3(1 - \cos \theta)$
$r = \dfrac{6}{1 - \cos \theta}$

Writing **In Exercises 29 and 30, use a graphing utility to find the points of intersection of the graphs of the polar equations. Watch the graphs as they are traced in the viewing window. Explain why the pole is not a point of intersection obtained by solving the equations simultaneously.**

29. $r = \cos \theta$
$r = 2 - 3 \sin \theta$

30. $r = 4 \sin \theta$
$r = 2(1 + \sin \theta)$

In Exercises 31–36, use a graphing utility to graph the polar equations and find the area of the given region.

31. Common interior of $r = 4\sin 2\theta$ and $r = 2$

32. Common interior of $r = 3(1 + \sin\theta)$ and $r = 3(1 - \sin\theta)$

33. Common interior of $r = 3 - 2\sin\theta$ and $r = -3 + 2\sin\theta$

34. Common interior of $r = 5 - 3\sin\theta$ and $r = 5 - 3\cos\theta$

35. Common interior of $r = 4\sin\theta$ and $r = 2$

36. Inside $r = 3\sin\theta$ and outside $r = 2 - \sin\theta$

In Exercises 37–40, find the area of the region.

37. Inside $r = a(1 + \cos\theta)$ and outside $r = a\cos\theta$

38. Inside $r = 2a\cos\theta$ and outside $r = a$

39. Common interior of $r = a(1 + \cos\theta)$ and $r = a\sin\theta$

40. Common interior of $r = a\cos\theta$ and $r = a\sin\theta$ where $a > 0$

41. ***Antenna Radiation*** The radiation from a transmitting antenna is not uniform in all directions. The intensity from a particular antenna is modeled by

$$r = a\cos^2\theta.$$

(a) Convert the polar equation to rectangular form.

(b) Use a graphing utility to graph the model for $a = 4$ and $a = 6$.

(c) Find the area of the geographical region between the two curves in part (b).

42. ***Area*** The area inside one or more of the three interlocking circles

$$r = 2a\cos\theta, \quad r = 2a\sin\theta, \quad \text{and} \quad r = a$$

is divided into seven regions. Find the area of each region.

43. ***Conjecture*** Find the area of the region enclosed by

$$r = a\cos(n\theta)$$

for $n = 1, 2, 3, \ldots$. Use the results to make a conjecture about the area enclosed by the function if n is even and if n is odd.

44. ***Area*** Sketch the strophoid

$$r = \sec\theta - 2\cos\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

Convert this equation to rectangular coordinates. Find the area enclosed by the loop.

In Exercises 45–48, find the length of the curve over the given interval.

	Polar Equation	*Interval*
45.	$r = a$	$0 \le \theta \le 2\pi$
46.	$r = 2a\cos\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
47.	$r = 1 + \sin\theta$	$0 \le \theta \le 2\pi$
48.	$r = 8(1 + \cos\theta)$	$0 \le \theta \le 2\pi$

In Exercises 49–54, use a graphing utility to graph the polar equation over the given interval. Use the integration capabilities of the graphing utility to approximate the length of the curve accurate to two decimal places.

49. $r = 2\theta, \quad 0 \le \theta \le \frac{\pi}{2}$

50. $r = \sec\theta, \quad 0 \le \theta \le \frac{\pi}{3}$

51. $r = \frac{1}{\theta}, \quad \pi \le \theta \le 2\pi$

52. $r = e^{\theta}, \quad 0 \le \theta \le \pi$

53. $r = \sin(3\cos\theta), \quad 0 \le \theta \le \pi$

54. $r = 2\sin(2\cos\theta), \quad 0 \le \theta \le \pi$

In Exercises 55–58, find the area of the surface formed by revolving the curve about the given line.

	Polar Equation	*Interval*	*Axis of Revolution*
55.	$r = 6\cos\theta$	$0 \le \theta \le \frac{\pi}{2}$	Polar axis
56.	$r = a\cos\theta$	$0 \le \theta \le \frac{\pi}{2}$	$\theta = \frac{\pi}{2}$
57.	$r = e^{a\theta}$	$0 \le \theta \le \frac{\pi}{2}$	$\theta = \frac{\pi}{2}$
58.	$r = a(1 + \cos\theta)$	$0 \le \theta \le \pi$	Polar axis

In Exercises 59 and 60, use the integration capabilities of a graphing utility to approximate to two decimal places the area of the surface formed by revolving the curve about the polar axis.

59. $r = 4\cos 2\theta, \quad 0 \le \theta \le \frac{\pi}{4}$

60. $r = \theta, \quad 0 \le \theta \le \pi$

Writing About Concepts

61. Give the integral formulas for area and arc length in polar coordinates.

62. Explain why finding points of intersection of polar graphs may require further analysis beyond solving two equations simultaneously.

63. Which integral yields the arc length of $r = 3(1 - \cos 2\theta)$? State why the other integrals are incorrect.

(a) $3\int_0^{2\pi} \sqrt{(1 - \cos 2\theta)^2 + 4\sin^2 2\theta}\, d\theta$

(b) $12\int_0^{\pi/4} \sqrt{(1 - \cos 2\theta)^2 + 4\sin^2 2\theta}\, d\theta$

(c) $3\int_0^{\pi} \sqrt{(1 - \cos 2\theta)^2 + 4\sin^2 2\theta}\, d\theta$

(d) $6\int_0^{\pi/2} \sqrt{(1 - \cos 2\theta)^2 + 4\sin^2 2\theta}\, d\theta$

64. Give the integral formulas for the area of the surface of revolution formed when the graph of $r = f(\theta)$ is revolved about (a) the x-axis and (b) the y-axis.

65. ***Surface Area of a Torus*** Find the surface area of the torus generated by revolving the circle given by $r = 2$ about the line $r = 5 \sec \theta$.

66. ***Surface Area of a Torus*** Find the surface area of the torus generated by revolving the circle given by $r = a$ about the line $r = b \sec \theta$, where $0 < a < b$.

67. ***Approximating Area*** Consider the circle $r = 8 \cos \theta$.

(a) Find the area of the circle.

(b) Complete the table giving the areas A of the sectors of the circle between $\theta = 0$ and the values of θ in the table.

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A							

(c) Use the table in part (b) to approximate the values of θ for which the sector of the circle composes $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ of the total area of the circle.

(d) Use a graphing utility to approximate, to two decimal places, the angles θ for which the sector of the circle composes $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ of the total area of the circle.

(e) Do the results of part (d) depend on the radius of the circle? Explain.

68. ***Approximate Area*** Consider the circle $r = 3 \sin \theta$.

(a) Find the area of the circle.

(b) Complete the table giving the areas A of the sectors of the circle between $\theta = 0$ and the values of θ in the table.

θ	0.2	0.4	0.6	0.8	1.0	1.2	1.4
A							

(c) Use the table in part (b) to approximate the values of θ for which the sector of the circle composes $\frac{1}{8}$, $\frac{1}{4}$, and $\frac{1}{2}$ of the total area of the circle.

(d) Use a graphing utility to approximate, to two decimal places, the angles θ for which the sector of the circle composes $\frac{1}{8}$, $\frac{1}{4}$, and $\frac{1}{2}$ of the total area of the circle.

69. What conic section does the following polar equation represent?

$$r = a \sin \theta + b \cos \theta$$

70. ***Area*** Find the area of the circle given by $r = \sin \theta + \cos \theta$. Check your result by converting the polar equation to rectangular form, then using the formula for the area of a circle.

71. ***Spiral of Archimedes*** The curve represented by the equation $r = a\theta$, where a is a constant, is called the spiral of Archimedes.

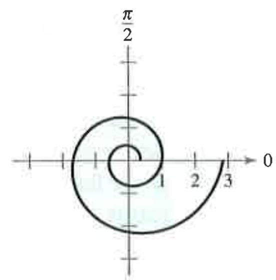
(a) Use a graphing utility to graph $r = \theta$, where $\theta \geq 0$. What happens to the graph of $r = a\theta$ as a increases? What happens if $\theta \leq 0$?

(b) Determine the points on the spiral $r = a\theta$ $(a > 0, \theta \geq 0)$, where the curve crosses the polar axis.

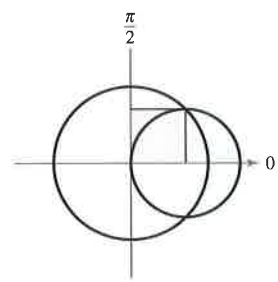
(c) Find the length of $r = \theta$ over the interval $0 \leq \theta \leq 2\pi$.

(d) Find the area under the curve $r = \theta$ for $0 \leq \theta \leq 2\pi$.

72. ***Logarithmic Spiral*** The curve represented by the equation $r = ae^{b\theta}$, where a and b are constants, is called a **logarithmic spiral.** The figure below shows the graph of $r = e^{\theta/6}$, $-2\pi \leq \theta \leq 2\pi$. Find the area of the shaded region.



73. The larger circle in the figure below is the graph of $r = 1$. Find the polar equation of the smaller circle such that the shaded regions are equal.



74. ***Folium of Descartes*** A curve called the **folium of Descartes** can be represented by the parametric equations

$$x = \frac{3t}{1 + t^3} \quad \text{and} \quad y = \frac{3t^2}{1 + t^3}.$$

(a) Convert the parametric equations to polar form.

(b) Sketch the graph of the polar equation from part (a).

(c) Use a graphing utility to approximate the area enclosed by the loop of the curve.

True or False? **In Exercises 75 and 76, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

75. If $f(\theta) > 0$ for all θ and $g(\theta) < 0$ for all θ, then the graphs of $r = f(\theta)$ and $r = g(\theta)$ do not intersect.

76. If $f(\theta) = g(\theta)$ for $\theta = 0$, $\pi/2$, and $3\pi/2$, then the graphs of $r = f(\theta)$ and $r = g(\theta)$ have at least four points of intersection.

77. Use the formula for the arc length of a curve in parametric form to derive the formula for the arc length of a polar curve.

Section 10.6 Polar Equations of Conics and Kepler's Laws

- **Analyze and write polar equations of conics.**
- **Understand and use Kepler's Laws of planetary motion.**

EXPLORATION

Graphing Conics Set a graphing utility to *polar* mode and enter polar equations of the form

$$r = \frac{a}{1 \pm b\cos\theta}$$

or

$$r = \frac{a}{1 \pm b\sin\theta}.$$

As long as $a \neq 0$, the graph should be a conic. Describe the values of a and b that produce parabolas. What values produce ellipses? What values produce hyperbolas?

Polar Equations of Conics

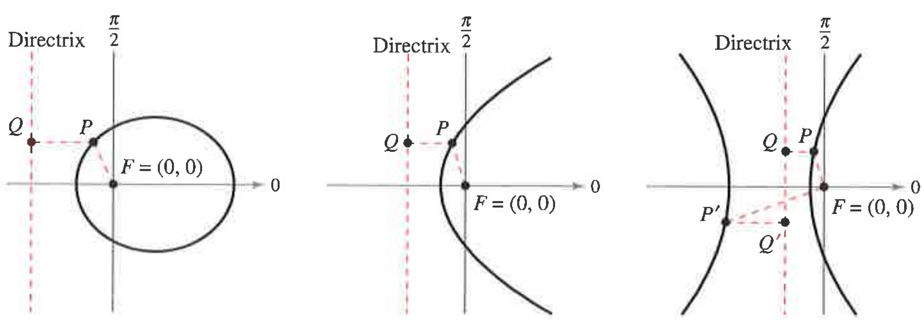
In this chapter you have seen that the rectangular equations of ellipses and hyperbolas take simple forms when the origin lies at their *centers*. As it happens, there are many important applications of conics in which it is more convenient to use one of the foci as the reference point (the origin) for the coordinate system. For example, the sun lies at a focus of Earth's orbit. Similarly, the light source of a parabolic reflector lies at its focus. In this section you will see that polar equations of conics take simple forms if one of the foci lies at the pole.

The following theorem uses the concept of *eccentricity*, as defined in Section 10.1, to classify the three basic types of conics. A proof of this theorem is given in Appendix A.

THEOREM 10.16 Classification of Conics by Eccentricity

Let F be a fixed point (*focus*) and D be a fixed line (*directrix*) in the plane. Let P be another point in the plane and let e (*eccentricity*) be the ratio of the distance between P and F to the distance between P and D. The collection of all points P with a given eccentricity is a conic.

1. The conic is an ellipse if $0 < e < 1$.
2. The conic is a parabola if $e = 1$.
3. The conic is a hyperbola if $e > 1$.



Ellipse: $0 < e < 1$, $\frac{PF}{PQ} < 1$

Parabola: $e = 1$, $PF = PQ$

Hyperbola: $e > 1$, $\frac{PF}{PQ} = \frac{P'F}{P'Q'} > 1$

Figure 10.58

In Figure 10.58, note that for each type of conic the pole corresponds to the fixed point (focus) given in the definition. The benefit of this location can be seen in the proof of the following theorem.

THEOREM 10.17 Polar Equations of Conics

The graph of a polar equation of the form

$$r = \frac{ed}{1 \pm e \cos \theta} \quad \text{or} \quad r = \frac{ed}{1 \pm e \sin \theta}$$

is a conic, where $e > 0$ is the eccentricity and $|d|$ is the distance between the focus at the pole and its corresponding directrix.

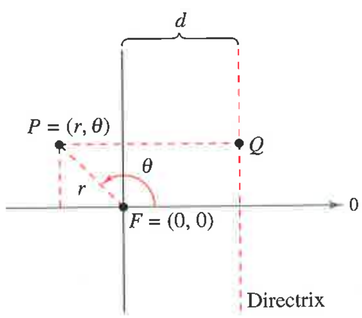


Figure 10.59

Proof The following is a proof for $r = ed/(1 + e \cos \theta)$ with $d > 0$. In Figure 10.59, consider a vertical directrix d units to the right of the focus $F = (0, 0)$. If $P = (r, \theta)$ is a point on the graph of $r = ed/(1 + e \cos \theta)$, the distance between P and the directrix can be shown to be

$$PQ = |d - x| = |d - r\cos\theta| = \left|\frac{r(1 + e\cos\theta)}{e} - r\cos\theta\right| = \left|\frac{r}{e}\right|.$$

Because the distance between P and the pole is simply $PF = |r|$, the ratio of PF to PQ is $PF/PQ = |r|/|r/e| = |e| = e$ and, by Theorem 10.16, the graph of the equation must be a conic. The proofs of the other cases are similar.

The four types of equations indicated in Theorem 10.17 can be classified as follows, where $d > 0$.

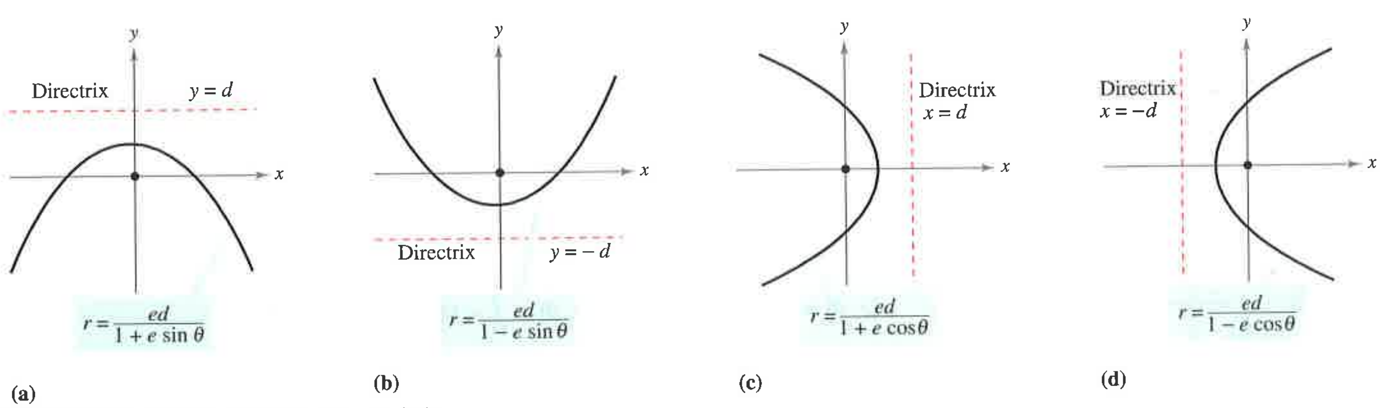
a. Horizontal directrix above the pole: $r = \dfrac{ed}{1 + e\sin\theta}$

b. Horizontal directrix below the pole: $r = \dfrac{ed}{1 - e\sin\theta}$

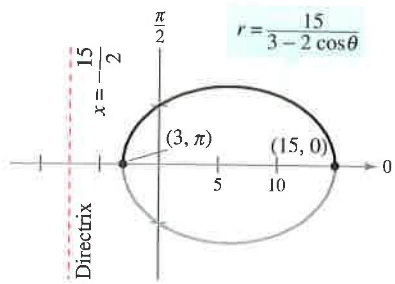
c. Vertical directrix to the right of the pole: $r = \dfrac{ed}{1 + e\cos\theta}$

d. Vertical directrix to the left of the pole: $r = \dfrac{ed}{1 - e\cos\theta}$

Figure 10.60 illustrates these four possibilities for a parabola.



The four types of polar equations for a parabola
Figure 10.60



The graph of the conic is an ellipse with $e = \frac{2}{3}$.
Figure 10.61

EXAMPLE 1 Determining a Conic from Its Equation

Sketch the graph of the conic given by $r = \dfrac{15}{3 - 2\cos\theta}$.

Solution To determine the type of conic, rewrite the equation as

$$r = \frac{15}{3 - 2\cos\theta} \qquad \text{Write original equation.}$$

$$= \frac{5}{1 - (2/3)\cos\theta}. \qquad \text{Divide numerator and denominator by 3.}$$

So, the graph is an ellipse with $e = \frac{2}{3}$. You can sketch the upper half of the ellipse by plotting points from $\theta = 0$ to $\theta = \pi$, as shown in Figure 10.61. Then, using symmetry with respect to the polar axis, you can sketch the lower half.

For the ellipse in Figure 10.61, the major axis is horizontal and the vertices lie at $(15, 0)$ and $(3, \pi)$. So, the length of the *major* axis is $2a = 18$. To find the length of the minor axis, you can use the equations $e = c/a$ and $b^2 = a^2 - c^2$ to conclude

$$b^2 = a^2 - c^2 = a^2 - (ea)^2 = a^2(1 - e^2). \qquad \text{Ellipse}$$

Because $e = \frac{2}{3}$, you have

$$b^2 = 9^2\left[1 - \left(\tfrac{2}{3}\right)^2\right] = 45$$

which implies that $b = \sqrt{45} = 3\sqrt{5}$. So, the length of the minor axis is $2b = 6\sqrt{5}$. A similar analysis for hyperbolas yields

$$b^2 = c^2 - a^2 = (ea)^2 - a^2 = a^2(e^2 - 1). \qquad \text{Hyperbola}$$

EXAMPLE 2 Sketching a Conic from Its Polar Equation

Sketch the graph of the polar equation $r = \dfrac{32}{3 + 5\sin\theta}$.

Solution Dividing the numerator and denominator by 3 produces

$$r = \frac{32/3}{1 + (5/3)\sin\theta}.$$

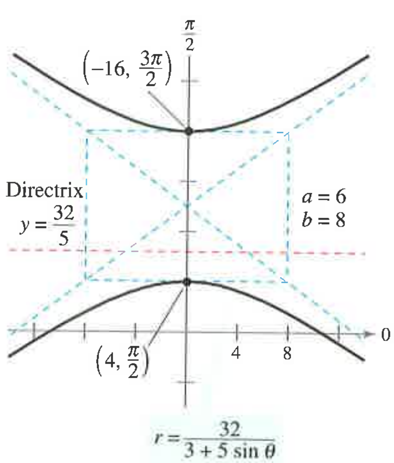
Because $e = \frac{5}{3} > 1$, the graph is a hyperbola. Because $d = \frac{32}{5}$, the directrix is the line $y = \frac{32}{5}$. The transverse axis of the hyperbola lies on the line $\theta = \pi/2$, and the vertices occur at

$$(r, \theta) = \left(4, \frac{\pi}{2}\right) \quad \text{and} \quad (r, \theta) = \left(-16, \frac{3\pi}{2}\right).$$

Because the length of the transverse axis is 12, you can see that $a = 6$. To find b, write

$$b^2 = a^2(e^2 - 1) = 6^2\left[\left(\frac{5}{3}\right)^2 - 1\right] = 64.$$

Therefore, $b = 8$. Finally, you can use a and b to determine the asymptotes of the hyperbola and obtain the sketch shown in Figure 10.62.



The graph of the conic is a hyperbola with $e = \frac{5}{3}$.
Figure 10.62

JOHANNES KEPLER (1571–1630)

Kepler formulated his three laws from the extensive data recorded by Danish astronomer Tycho Brahe, and from direct observation of the orbit of Mars.

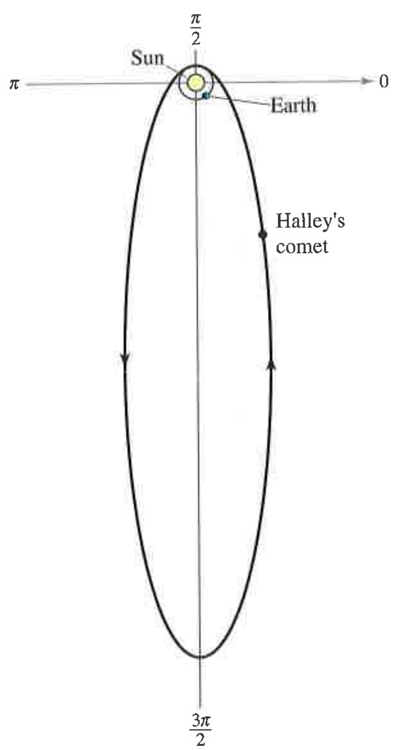


Figure 10.63

Kepler's Laws

Kepler's Laws, named after the German astronomer Johannes Kepler, can be used to describe the orbits of the planets about the sun.

1. Each planet moves in an elliptical orbit with the sun as a focus.
2. A ray from the sun to the planet sweeps out equal areas of the ellipse in equal times.
3. The square of the period is proportional to the cube of the mean distance between the planet and the sun.*

Although Kepler derived these laws empirically, they were later validated by Newton. In fact, Newton was able to show that each law can be deduced from a set of universal laws of motion and gravitation that govern the movement of all heavenly bodies, including comets and satellites. This is shown in the next example, involving the comet named after the English mathematician and physicist Edmund Halley (1656–1742).

EXAMPLE 3 Halley's Comet

Halley's comet has an elliptical orbit with the sun at one focus and has an eccentricity of $e \approx 0.967$. The length of the major axis of the orbit is approximately 35.88 astronomical units. (An astronomical unit is defined to be the mean distance between Earth and the sun, 93 million miles.) Find a polar equation for the orbit. How close does Halley's comet come to the sun?

Solution Using a vertical axis, you can choose an equation of the form

$$r = \frac{ed}{(1 + e \sin \theta)}.$$

Because the vertices of the ellipse occur when $\theta = \pi/2$ and $\theta = 3\pi/2$, you can determine the length of the major axis to be the sum of the r-values of the vertices, as shown in Figure 10.63. That is,

$$2a = \frac{0.967d}{1 + 0.967} + \frac{0.967d}{1 - 0.967}$$

$$35.88 \approx 27.79d. \qquad 2a \approx 35.88$$

So, $d \approx 1.204$ and $ed \approx (0.967)(1.204) \approx 1.164$. Using this value in the equation produces

$$r = \frac{1.164}{1 + 0.967 \sin \theta}$$

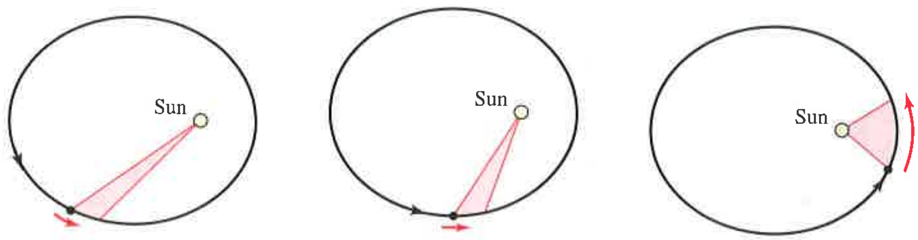
where r is measured in astronomical units. To find the closest point to the sun (the focus), you can write $c = ea \approx (0.967)(17.94) \approx 17.35$. Because c is the distance between the focus and the center, the closest point is

$$\begin{aligned} a - c &\approx 17.94 - 17.35 \\ &\approx 0.59 \text{ AU} \\ &\approx 55{,}000{,}000 \text{ miles} \end{aligned}$$

---

* *If Earth is used as a reference with a period of 1 year and a distance of 1 astronomical unit, the proportionality constant is 1. For example, because Mars has a mean distance to the sun of $D = 1.524$ AU, its period P is given by $D^3 = P^2$. So, the period for Mars is $P = 1.88$.*

Kepler's Second Law states that as a planet moves about the sun, a ray from the sun to the planet sweeps out equal areas in equal times. This law can also be applied to comets or asteroids with elliptical orbits. For example, Figure 10.64 shows the orbit of the asteroid Apollo about the sun. Applying Kepler's Second Law to this asteroid, you know that the closer it is to the sun, the greater its velocity, because a short ray must be moving quickly to sweep out as much area as a long ray.



A ray from the sun to the asteroid sweeps out equal areas in equal times.
Figure 10.64

EXAMPLE 4 The Asteroid Apollo

The asteroid Apollo has a period of 661 Earth days, and its orbit is approximated by the ellipse

$$r = \frac{1}{1 + (5/9)\cos\theta} = \frac{9}{9 + 5\cos\theta}$$

where r is measured in astronomical units. How long does it take Apollo to move from the position given by $\theta = -\pi/2$ to $\theta = \pi/2$, as shown in Figure 10.65?

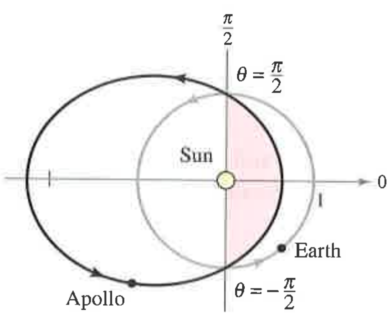


Figure 10.65

Solution Begin by finding the area swept out as θ increases from $-\pi/2$ to $\pi/2$.

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta \qquad \text{Formula for area of a polar graph}$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(\frac{9}{9 + 5\cos\theta}\right)^2 d\theta$$

Using the substitution $u = \tan(\theta/2)$, as discussed in Section 8.6, you obtain

$$A = \frac{81}{112}\left[\frac{-5\sin\theta}{9 + 5\cos\theta} + \frac{18}{\sqrt{56}}\arctan\frac{\sqrt{56}\tan(\theta/2)}{14}\right]_{-\pi/2}^{\pi/2} \approx 0.90429.$$

Because the major axis of the ellipse has length $2a = 81/28$ and the eccentricity is $e = 5/9$, you can determine that $b = a\sqrt{1 - e^2} = 9/\sqrt{56}$. So, the area of the ellipse is

$$\text{Area of ellipse} = \pi ab = \pi\left(\frac{81}{56}\right)\left(\frac{9}{\sqrt{56}}\right) \approx 5.46507.$$

Because the time required to complete the orbit is 661 days, you can apply Kepler's Second Law to conclude that the time t required to move from the position $\theta = -\pi/2$ to $\theta = \pi/2$ is given by

$$\frac{t}{661} = \frac{\text{area of elliptical segment}}{\text{area of ellipse}} \approx \frac{0.90429}{5.46507}$$

which implies that $t \approx 109$ days.

Exercises for Section 10.6

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Graphical Reasoning **In Exercises 1–4, use a graphing utility to graph the polar equation when (a) $e = 1$, (b) $e = 0.5$, and (c) $e = 1.5$. Identify the conic.**

1. $r = \dfrac{2e}{1 + e\cos\theta}$

2. $r = \dfrac{2e}{1 - e\cos\theta}$

3. $r = \dfrac{2e}{1 - e\sin\theta}$

4. $r = \dfrac{2e}{1 + e\sin\theta}$

5. ***Writing*** Consider the polar equation

$$r = \frac{4}{1 + e\sin\theta}.$$

(a) Use a graphing utility to graph the equation for $e = 0.1$, $e = 0.25$, $e = 0.5$, $e = 0.75$, and $e = 0.9$. Identify the conic and discuss the change in its shape as $e \to 1^-$ and $e \to 0^+$.

(b) Use a graphing utility to graph the equation for $e = 1$. Identify the conic.

(c) Use a graphing utility to graph the equation for $e = 1.1$, $e = 1.5$, and $e = 2$. Identify the conic and discuss the change in its shape as $e \to 1^+$ and $e \to \infty$.

6. Consider the polar equation

$$r = \frac{4}{1 - 0.4\cos\theta}.$$

(a) Identify the conic without graphing the equation.

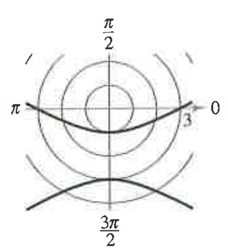
(b) Without graphing the following polar equations, describe how each differs from the polar equation above.

$$r = \frac{4}{1 + 0.4\cos\theta}, \quad r = \frac{4}{1 - 0.4\sin\theta}$$

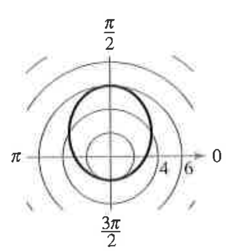
(c) Verify the results of part (b) graphically.

In Exercises 7–12, match the polar equation with the correct graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

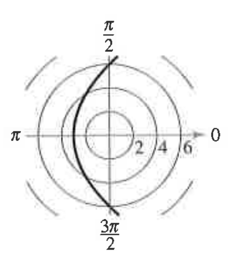
(a)



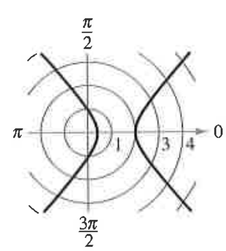
(b)



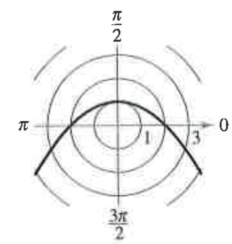
(c)



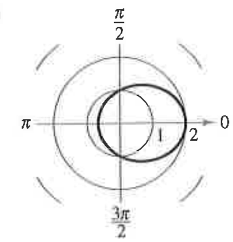
(d)



(e)



(f)



7. $r = \dfrac{6}{1 - \cos\theta}$

8. $r = \dfrac{2}{2 - \cos\theta}$

9. $r = \dfrac{3}{1 - 2\sin\theta}$

10. $r = \dfrac{2}{1 + \sin\theta}$

11. $r = \dfrac{6}{2 - \sin\theta}$

12. $r = \dfrac{2}{2 + 3\cos\theta}$

In Exercises 13–22, find the eccentricity and the distance from the pole to the directrix of the conic. Then sketch and identify the graph. Use a graphing utility to confirm your results.

13. $r = \dfrac{-1}{1 - \sin\theta}$

14. $r = \dfrac{6}{1 + \cos\theta}$

15. $r = \dfrac{6}{2 + \cos\theta}$

16. $r = \dfrac{5}{5 + 3\sin\theta}$

17. $r(2 + \sin\theta) = 4$

18. $r(3 - 2\cos\theta) = 6$

19. $r = \dfrac{5}{-1 + 2\cos\theta}$

20. $r = \dfrac{-6}{3 + 7\sin\theta}$

21. $r = \dfrac{3}{2 + 6\sin\theta}$

22. $r = \dfrac{4}{1 + 2\cos\theta}$

In Exercises 23–26, use a graphing utility to graph the polar equation. Identify the graph.

23. $r = \dfrac{3}{-4 + 2\sin\theta}$

24. $r = \dfrac{-3}{2 + 4\sin\theta}$

25. $r = \dfrac{-1}{1 - \cos\theta}$

26. $r = \dfrac{2}{2 + 3\sin\theta}$

In Exercises 27–30, use a graphing utility to graph the conic. Describe how the graph differs from that in the indicated exercise.

27. $r = \dfrac{-1}{1 - \sin(\theta - \pi/4)}$ (See Exercise 13.)

28. $r = \dfrac{6}{1 + \cos(\theta - \pi/3)}$ (See Exercise 14.)

29. $r = \dfrac{6}{2 + \cos(\theta + \pi/6)}$ (See Exercise 15.)

30. $r = \dfrac{-6}{3 + 7\sin(\theta + 2\pi/3)}$ (See Exercise 20.)

31. Write the equation for the ellipse rotated $\pi/4$ radian clockwise from the ellipse

$$r = \frac{5}{5 + 3\cos\theta}.$$

32. Write the equation for the parabola rotated $\pi/6$ radian counterclockwise from the parabola

$$r = \frac{2}{1 + \sin\theta}.$$

In Exercises 33–44, find a polar equation for the conic with its focus at the pole. (For convenience, the equation for the directrix is given in rectangular form.)

	Conic	*Eccentricity*	*Directrix*
33.	Parabola	$e = 1$	$x = -1$
34.	Parabola	$e = 1$	$y = 1$
35.	Ellipse	$e = \frac{1}{2}$	$y = 1$
36.	Ellipse	$e = \frac{3}{4}$	$y = -2$
37.	Hyperbola	$e = 2$	$x = 1$
38.	Hyperbola	$e = \frac{3}{2}$	$x = -1$

	Conic	*Vertex or Vertices*
39.	Parabola	$\left(1, -\frac{\pi}{2}\right)$
40.	Parabola	$(5, \pi)$
41.	Ellipse	$(2, 0),\ (8, \pi)$
42.	Ellipse	$\left(2, \frac{\pi}{2}\right),\ \left(4, \frac{3\pi}{2}\right)$
43.	Hyperbola	$\left(1, \frac{3\pi}{2}\right),\ \left(9, \frac{3\pi}{2}\right)$
44.	Hyperbola	$(2, 0),\ (10, 0)$

Writing About Concepts

45. Classify the conics by their eccentricities.

46. Explain how the graph of each conic differs from the graph of $r = \dfrac{4}{1 + \sin\theta}$.

(a) $r = \dfrac{4}{1 - \cos\theta}$ (b) $r = \dfrac{4}{1 - \sin\theta}$

(c) $r = \dfrac{4}{1 + \cos\theta}$ (d) $r = \dfrac{4}{1 - \sin(\theta - \pi/4)}$

47. Identify each conic.

(a) $r = \dfrac{5}{1 - 2\cos\theta}$ (b) $r = \dfrac{5}{10 - \sin\theta}$

(c) $r = \dfrac{5}{3 - 3\cos\theta}$ (d) $r = \dfrac{5}{1 - 3\sin(\theta - \pi/4)}$

48. Describe what happens to the distance between the directrix and the center of an ellipse if the foci remain fixed and e approaches 0.

49. Show that the polar equation for $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$r^2 = \frac{b^2}{1 - e^2\cos^2\theta}.$$ Ellipse

50. Show that the polar equation for $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$r^2 = \frac{-b^2}{1 - e^2\cos^2\theta}.$$ Hyperbola

In Exercises 51–54, use the results of Exercises 49 and 50 to write the polar form of the equation of the conic.

51. Ellipse: focus at $(4, 0)$; vertices at $(5, 0)$, $(5, \pi)$

52. Hyperbola: focus at $(5, 0)$; vertices at $(4, 0)$, $(4, \pi)$

53. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

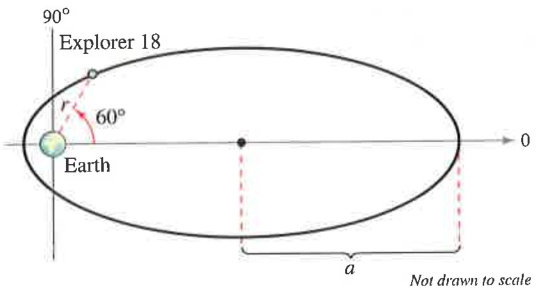
54. $\dfrac{x^2}{4} + y^2 = 1$

In Exercises 55 and 56, use the integration capabilities of a graphing utility to approximate to two decimal places the area of the region bounded by the graph of the polar equation.

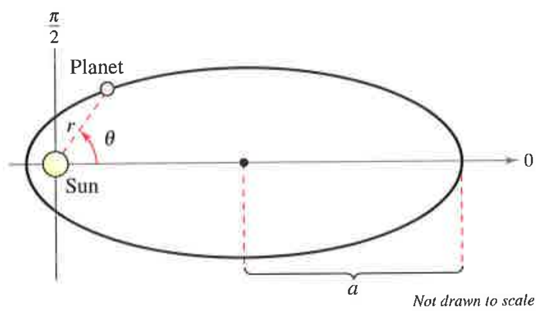
55. $r = \dfrac{3}{2 - \cos\theta}$

56. $r = \dfrac{2}{3 - 2\sin\theta}$

57. ***Explorer 18*** On November 27, 1963, the United States launched Explorer 18. Its low and high points above the surface of Earth were approximately 119 miles and 123,000 miles (see figure). The center of Earth is the focus of the orbit. Find the polar equation for the orbit and find the distance between the surface of Earth and the satellite when $\theta = 60°$. (Assume that the radius of Earth is 4000 miles.)



58. ***Planetary Motion*** The planets travel in elliptical orbits with the sun as a focus, as shown in the figure.



(a) Show that the polar equation of the orbit is given by

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta}$$

where e is the eccentricity.

(b) Show that the minimum distance (*perihelion*) from the sun to the planet is $r = a(1 - e)$ and the maximum distance (*aphelion*) is $r = a(1 + e)$.

In Exercises 59–62, use Exercise 58 to find the polar equation of the elliptical orbit of the planet, and the perihelion and aphelion distances.

59. Earth $a = 1.496 \times 10^8$ kilometers
$e = 0.0167$

60. Saturn $a = 1.427 \times 10^9$ kilometers
$e = 0.0542$

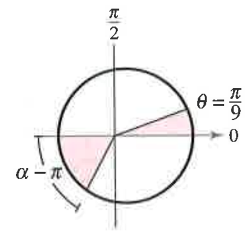
61. Pluto $a = 5.906 \times 10^9$ kilometers
$e = 0.2488$

62. Mercury $a = 5.791 \times 10^7$ kilometers
$e = 0.2056$

63. ***Planetary Motion*** In Exercise 61, the polar equation for the elliptical orbit of Pluto was found. Use the equation and a computer algebra system to perform each of the following.

(a) Approximate the area swept out by a ray from the sun to the planet as θ increases from 0 to $\pi/9$. Use this result to determine the number of years for the planet to move through this arc if the period of one revolution around the sun is 248 years.

(b) By trial and error, approximate the angle α such that the area swept out by a ray from the sun to the planet as θ increases from π to α equals the area found in part (a) (see figure). Does the ray sweep through a larger or smaller angle than in part (a) to generate the same area? Why is this the case?



(c) Approximate the distances the planet traveled in parts (a) and (b). Use these distances to approximate the average number of kilometers per year the planet traveled in the two cases.

64. ***Comet Hale-Bopp*** The comet Hale-Bopp has an elliptical orbit with the sun at one focus and has an eccentricity of $e \approx 0.995$. The length of the major axis of the orbit is approximately 250 astronomical units.

(a) Find the length of its minor axis.

(b) Find a polar equation for the orbit.

(c) Find the perihelion and aphelion distances.

In Exercises 65 and 66, let r_0 represent the distance from the focus to the nearest vertex, and let r_1 represent the distance from the focus to the farthest vertex.

65. Show that the eccentricity of an ellipse can be written as

$$e = \frac{r_1 - r_0}{r_1 + r_0}. \text{ Then show that } \frac{r_1}{r_0} = \frac{1 + e}{1 - e}.$$

66. Show that the eccentricity of a hyperbola can be written as

$$e = \frac{r_1 + r_0}{r_1 - r_0}. \text{ Then show that } \frac{r_1}{r_0} = \frac{e + 1}{e - 1}.$$

In Exercises 67 and 68, show that the graphs of the given equations intersect at right angles.

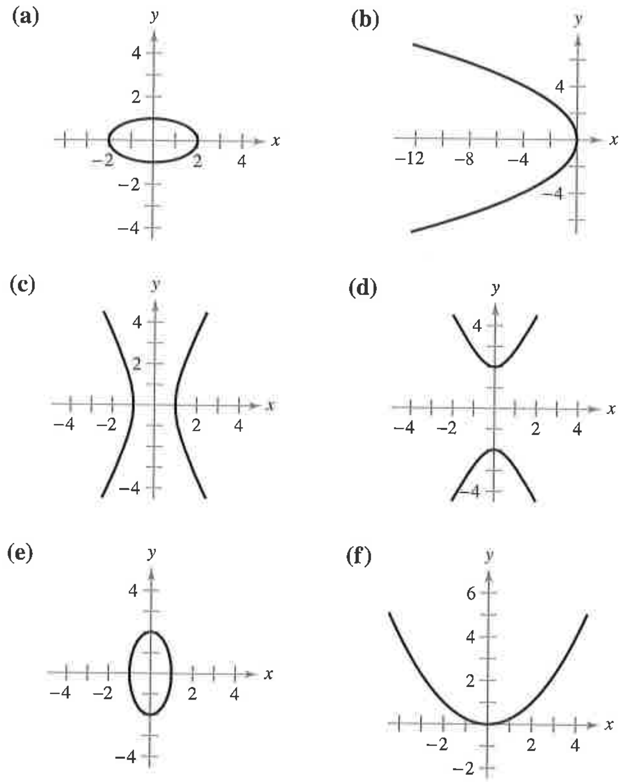
67. $r = \dfrac{ed}{1 + \sin\theta}$ and $r = \dfrac{ed}{1 - \sin\theta}$

68. $r = \dfrac{c}{1 + \cos\theta}$ and $r = \dfrac{d}{1 - \cos\theta}$

Review Exercises for Chapter 10

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, match the equation with the correct graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]



1. $4x^2 + y^2 = 4$
2. $4x^2 - y^2 = 4$
3. $y^2 = -4x$
4. $y^2 - 4x^2 = 4$
5. $x^2 + 4y^2 = 4$
6. $x^2 = 4y$

In Exercises 7–12, analyze the equation and sketch its graph. Use a graphing utility to confirm your results.

7. $16x^2 + 16y^2 - 16x + 24y - 3 = 0$
8. $y^2 - 12y - 8x + 20 = 0$
9. $3x^2 - 2y^2 + 24x + 12y + 24 = 0$
10. $4x^2 + y^2 - 16x + 15 = 0$
11. $3x^2 + 2y^2 - 12x + 12y + 29 = 0$
12. $4x^2 - 4y^2 - 4x + 8y - 11 = 0$

In Exercises 13 and 14, find an equation of the parabola.

13. Vertex: $(0, 2)$; directrix: $x = -3$
14. Vertex: $(4, 2)$; focus: $(4, 0)$

In Exercises 15 and 16, find an equation of the ellipse.

15. Vertices: $(-3, 0), (7, 0)$; foci: $(0, 0), (4, 0)$
16. Center: $(0, 0)$; solution points: $(1, 2)$, $(2, 0)$

In Exercises 17 and 18, find an equation of the hyperbola.

17. Vertices: $(\pm 4, 0)$; foci: $(\pm 6, 0)$
18. Foci: $(0, \pm 8)$; asymptotes: $y = \pm 4x$

In Exercises 19 and 20, use a graphing utility to approximate the perimeter of the ellipse.

19. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$
20. $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$

21. A line is tangent to the parabola $y = x^2 - 2x + 2$ and perpendicular to the line $y = x - 2$. Find the equation of the line.
22. A line is tangent to the parabola $3x^2 + y = x - 6$ and perpendicular to the line $2x + y = 5$. Find the equation of the line.
23. ***Satellite Antenna*** A cross section of a large parabolic antenna is modeled by the graph of

$$y = \frac{x^2}{200}, \quad -100 \le x \le 100.$$

The receiving and transmitting equipment is positioned at the focus.

(a) Find the coordinates of the focus.

(b) Find the surface area of the antenna.

24. ***Fire Truck*** Consider a fire truck with a water tank 16 feet long whose vertical cross sections are ellipses modeled by the equation

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

(a) Find the volume of the tank.

(b) Find the force on the end of the tank when it is full of water. (The density of water is 62.4 pounds per cubic foot.)

(c) Find the depth of the water in the tank if it is $\frac{3}{4}$ full (by volume) and the truck is on level ground.

(d) Approximate the tank's surface area.

In Exercises 25–30, sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.

25. $x = 1 + 4t, \ y = 2 - 3t$
26. $x = t + 4, \ y = t^2$
27. $x = 6 \cos \theta, \ y = 6 \sin \theta$
28. $x = 3 + 3 \cos \theta, \ y = 2 + 5 \sin \theta$
29. $x = 2 + \sec \theta, \ y = 3 + \tan \theta$
30. $x = 5 \sin^3 \theta, \ y = 5 \cos^3 \theta$

In Exercises 31–34, find a parametric representation of the line or conic.

31. Line: passes through $(-2, 6)$ and $(3, 2)$

32. Circle: center at $(5, 3)$; radius 2

33. Ellipse: center at $(-3, 4)$; horizontal major axis of length 8 and minor axis of length 6

34. Hyperbola: vertices at $(0, \pm 4)$; foci at $(0, \pm 5)$

35. ***Rotary Engine*** The rotary engine was developed by Felix Wankel in the 1950s. It features a rotor, which is a modified equilateral triangle. The rotor moves in a chamber that, in two dimensions, is an epitrochoid. Use a graphing utility to graph the chamber modeled by the parametric equations.

$x = \cos 3\theta + 5 \cos \theta$

and

$y = \sin 3\theta + 5 \sin \theta.$

36. ***Serpentine Curve*** Consider the parametric equations $x = 2 \cot \theta$ and $y = 4 \sin \theta \cos \theta$, $0 < \theta < \pi$.

(a) Use a graphing utility to graph the curve.

(b) Eliminate the parameter to show that the rectangular equation of the serpentine curve is $(4 + x^2)y = 8x$.

In Exercises 37–46, (a) find dy/dx and all points of horizontal tangency, (b) eliminate the parameter where possible, and (c) sketch the curve represented by the parametric equations.

37. $x = 1 + 4t, \quad y = 2 - 3t$

38. $x = t + 4, \quad y = t^2$

39. $x = \dfrac{1}{t}, \quad y = 2t + 3$

40. $x = \dfrac{1}{t}, \quad y = t^2$

41. $x = \dfrac{1}{2t + 1}$, $y = \dfrac{1}{t^2 - 2t}$

42. $x = 2t - 1$, $y = \dfrac{1}{t^2 - 2t}$

43. $x = 3 + 2 \cos \theta$, $y = 2 + 5 \sin \theta$

44. $x = 6 \cos \theta$, $y = 6 \sin \theta$

45. $x = \cos^3 \theta$, $y = 4 \sin^3 \theta$

46. $x = e^t$, $y = e^{-t}$

In Exercises 47–50, find all points (if any) of horizontal and vertical tangency to the curve. Use a graphing utility to confirm your results.

47. $x = 4 - t, \quad y = t^2$

48. $x = t + 2, \quad y = t^3 - 2t$

49. $x = 2 + 2 \sin \theta, \quad y = 1 + \cos \theta$

50. $x = 2 - 2 \cos \theta, \quad y = 2 \sin 2\theta$

In Exercises 51 and 52, (a) use a graphing utility to graph the curve represented by the parametric equations, (b) use a graphing utility to find $dx/d\theta$, $dy/d\theta$, and dy/dx for $\theta = \pi/6$, and (c) use a graphing utility to graph the tangent line to the curve when $\theta = \pi/6$.

51. $x = \cot \theta$, $y = \sin 2\theta$

52. $x = 2\theta - \sin \theta$, $y = 2 - \cos \theta$

Arc Length **In Exercises 53 and 54, find the arc length of the curve on the given interval.**

53. $x = r(\cos \theta + \theta \sin \theta)$, $y = r(\sin \theta - \theta \cos \theta)$, $0 \le \theta \le \pi$

54. $x = 6 \cos \theta$, $y = 6 \sin \theta$, $0 \le \theta \le \pi$

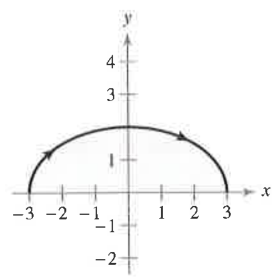
Surface Area **In Exercises 55 and 56, find the area of the surface generated by revolving the curve about (a) the x-axis and (b) the y-axis.**

55. $x = t, \quad y = 3t, \quad 0 \le t \le 2$

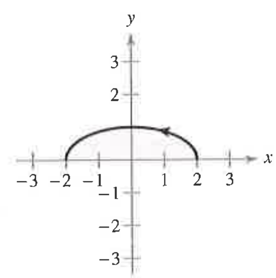
56. $x = 2 \cos \theta, \quad y = 2 \sin \theta, \quad 0 \le \theta \le \dfrac{\pi}{2}$

Area **In Exercises 57 and 58, find the area of the region.**

57. $x = 3 \sin \theta$, $y = 2 \cos \theta$, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$



58. $x = 2 \cos \theta$, $y = \sin \theta$, $0 \le \theta \le \pi$



In Exercises 59–62, plot the point in polar coordinates and find the corresponding rectangular coordinates of the point.

59. $\left(3, \dfrac{\pi}{2}\right)$

60. $\left(-4, \dfrac{11\pi}{6}\right)$

61. $(\sqrt{3}, 1.56)$

62. $(-2, -2.45)$

In Exercises 63 and 64, the rectangular coordinates of a point are given. Plot the point and find *two* sets of polar coordinates of the point for $0 \le \theta < 2\pi$.

63. $(4, -4)$

64. $(-1, 3)$

In Exercises 65–72, convert the polar equation to rectangular form.

65. $r = 3\cos\theta$
66. $r = 10$
67. $r = -2(1 + \cos\theta)$
68. $r = \dfrac{1}{2 - \cos\theta}$
69. $r^2 = \cos 2\theta$
70. $r = 4\sec\left(\theta - \dfrac{\pi}{3}\right)$
71. $r = 4\cos 2\theta \sec\theta$
72. $\theta = \dfrac{3\pi}{4}$

In Exercises 73–76, convert the rectangular equation to polar form.

73. $(x^2 + y^2)^2 = ax^2y$
74. $x^2 + y^2 - 4x = 0$
75. $x^2 + y^2 = a^2\left(\arctan\dfrac{y}{x}\right)^2$
76. $(x^2 + y^2)\left(\arctan\dfrac{y}{x}\right)^2 = a^2$

In Exercises 77–88, sketch a graph of the polar equation.

77. $r = 4$
78. $\theta = \dfrac{\pi}{12}$
79. $r = -\sec\theta$
80. $r = 3\csc\theta$
81. $r = -2(1 + \cos\theta)$
82. $r = 3 - 4\cos\theta$
83. $r = 4 - 3\cos\theta$
84. $r = 2\theta$
85. $r = -3\cos 2\theta$
86. $r = \cos 5\theta$
87. $r^2 = 4\sin^2 2\theta$
88. $r^2 = \cos 2\theta$

In Exercises 89–92, use a graphing utility to graph the polar equation.

89. $r = \dfrac{3}{\cos(\theta - \pi/4)}$
90. $r = 2\sin\theta\cos^2\theta$
91. $r = 4\cos 2\theta\sec\theta$
92. $r = 4(\sec\theta - \cos\theta)$

In Exercises 93 and 94, (a) find the tangents at the pole, (b) find all points of vertical and horizontal tangency, and (c) use a graphing utility to graph the polar equation and draw a tangent line to the graph for $\theta = \pi/6$.

93. $r = 1 - 2\cos\theta$
94. $r^2 = 4\sin 2\theta$

95. Find the angle between the circle $r = 3\sin\theta$ and the limaçon $r = 4 - 5\sin\theta$ at the point of intersection $(3/2, \pi/6)$.

96. ***True or False?*** There is a unique polar coordinate representation for each point in the plane. Explain.

In Exercises 97 and 98, show that the graphs of the polar equations are orthogonal at the points of intersection. Use a graphing utility to confirm your results graphically.

97. $r = 1 + \cos\theta$
$r = 1 - \cos\theta$
98. $r = a\sin\theta$
$r = a\cos\theta$

In Exercises 99–102, find the area of the region.

99. Interior of $r = 2 + \cos\theta$
100. Interior of $r = 5(1 - \sin\theta)$
101. Interior of $r^2 = 4\sin 2\theta$
102. Common interior of $r = 4\cos\theta$ and $r = 2$

In Exercises 103–106, use a graphing utility to graph the polar equation. Set up an integral for finding the area of the given region and use the integration capabilities of a graphing utility to approximate the integral accurate to two decimal places.

103. Interior of $r = \sin\theta\cos^2\theta$
104. Interior of $r = 4\sin 3\theta$
105. Common interior of $r = 3$ and $r^2 = 18\sin 2\theta$
106. Region bounded by the polar axis and $r = e^{\theta}$ for $0 \le \theta \le \pi$

In Exercises 107 and 108, find the length of the curve over the given interval.

	Polar Equation	*Interval*
107.	$r = a(1 - \cos\theta)$	$0 \le \theta \le \pi$
108.	$r = a\cos 2\theta$	$-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

In Exercises 109 and 110, write an integral that represents the area of the surface formed by revolving the curve about the given line. Use a graphing utility to approximate the integral.

	Polar Equation	*Interval*	*Axis of Revolution*
109.	$r = 1 + 4\cos\theta$	$0 \le \theta \le \dfrac{\pi}{2}$	Polar axis
110.	$r = 2\sin\theta$	$0 \le \theta \le \dfrac{\pi}{2}$	$\theta = \dfrac{\pi}{2}$

In Exercises 111–116, sketch and identify the graph. Use a graphing utility to confirm your results.

111. $r = \dfrac{2}{1 - \sin\theta}$
112. $r = \dfrac{2}{1 + \cos\theta}$
113. $r = \dfrac{6}{3 + 2\cos\theta}$
114. $r = \dfrac{4}{5 - 3\sin\theta}$
115. $r = \dfrac{4}{2 - 3\sin\theta}$
116. $r = \dfrac{8}{2 - 5\cos\theta}$

In Exercises 117–122, find a polar equation for the line or conic with its focus at the pole.

117. Circle
Center: $(5, \pi/2)$
Solution point: $(0, 0)$

118. Line
Solution point: $(0, 0)$
Slope: $\sqrt{3}$

119. Parabola
Vertex: $(2, \pi)$

120. Parabola
Vertex: $(2, \pi/2)$

121. Ellipse
Vertices: $(5, 0)$, $(1, \pi)$

122. Hyperbola
Vertices: $(1, 0)$, $(7, 0)$

P.S. Problem Solving

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

1. Consider the parabola $x^2 = 4y$ and the focal chord $y = \frac{3}{4}x + 1$.

(a) Sketch the graph of the parabola and the focal chord.

(b) Show that the tangent lines to the parabola at the endpoints of the focal chord intersect at right angles.

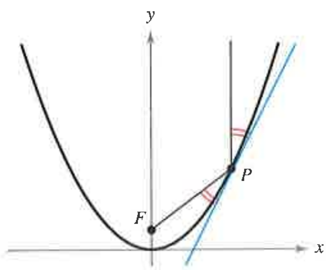
(c) Show that the tangent lines to the parabola at the endpoints of the focal chord intersect on the directrix of the parabola.

2. Consider the parabola $x^2 = 4py$ and one of its focal chords.

(a) Show that the tangent lines to the parabola at the endpoints of the focal chord intersect at right angles.

(b) Show that the tangent lines to the parabola at the endpoints of the focal chord intersect on the directrix of the parabola.

3. Prove Theorem 10.2, Reflective Property of a Parabola, as shown in the figure.



4. Consider the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

with foci F_1 and F_2, as shown in the figure. Let T be the tangent line at a point M on the hyperbola. Show that incoming rays of light aimed at one focus are reflected by a hyperbolic mirror toward the other focus.

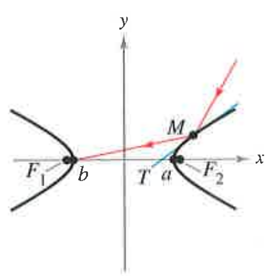


Figure for 4

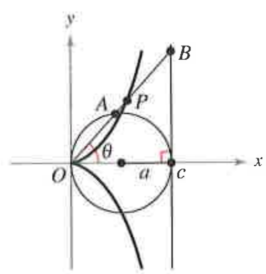


Figure for 5

5. Consider a circle of radius a tangent to the y-axis and the line $x = 2a$, as shown in the figure. Let A be the point where the segment OB intersects the circle. The **cissoid of Diocles** consists of all points P such that $OP = AB$.

(a) Find a polar equation of the cissoid.

(b) Find a set of parametric equations for the cissoid that does not contain trigonometric functions.

(c) Find a rectangular equation of the cissoid.

6. Consider the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$, with eccentricity $e = c/a$.

(a) Show that the area of the region is πab.

(b) Show that the solid (oblate spheroid) generated by revolving the region about the minor axis of the ellipse has a volume $V = 4\pi^2 b/3$ and a surface area of

$$S = 2\pi a^2 + \pi\left(\frac{b^2}{e}\right)\ln\left(\frac{1+e}{1-e}\right).$$

(c) Show that the solid (prolate spheroid) generated by revolving the region about the major axis of the ellipse has a volume of $V = 4\pi ab^2/3$ and a surface area of

$$S = 2\pi b^2 + 2\pi\left(\frac{ab}{e}\right)\arcsin e.$$

7. The curve given by the parametric equations

$$x(t) = \frac{1-t^2}{1+t^2} \quad \text{and} \quad y(t) = \frac{t(1-t^2)}{1+t^2}$$

is called a **strophoid.**

(a) Find a rectangular equation of the strophoid.

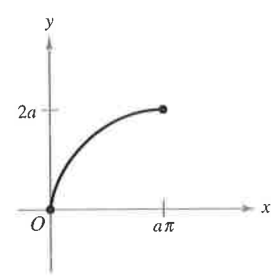
(b) Find a polar equation of the strophoid.

(c) Sketch a graph of the strophoid.

(d) Find the equations of the two tangent lines at the origin.

(e) Find the points on the graph where the tangent lines are horizontal.

8. Find a rectangular equation of the portion of the cycloid given by the parametric equations $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$, $0 \le \theta \le \pi$, as shown in the figure.



9. Consider the **cornu spiral** given by

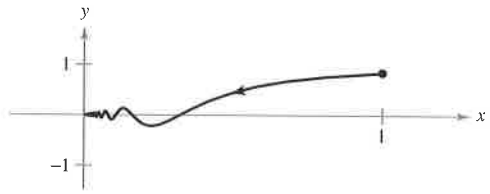
$$x(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du \quad \text{and} \quad y(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du.$$

(a) Use a graphing utility to graph the spiral over the interval $-\pi \le t \le \pi$.

(b) Show that the cornu spiral is symmetric with respect to the origin.

(c) Find the length of the cornu spiral from $t = 0$ to $t = a$. What is the length of the spiral from $t = -\pi$ to $t = \pi$?

10. A particle is moving along the path described by the parametric equations $x = 1/t$ and $y = \sin t/t$, for $1 \leq t < \infty$, as shown in the figure. Find the length of this path.



11. Let a and b be positive constants. Find the area of the region in the first quadrant bounded by the graph of the polar equation

$$r = \frac{ab}{(a \sin \theta + b \cos \theta)}, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

12. Consider the right triangle shown in the figure.

(a) Show that the area of the triangle is $A(\alpha) = \dfrac{1}{2}\displaystyle\int_0^\alpha \sec^2 \theta \, d\theta.$

(b) Show that $\tan \alpha = \displaystyle\int_0^\alpha \sec^2 \theta \, d\theta.$

(c) Use part (b) to derive the formula for the derivative of the tangent function.

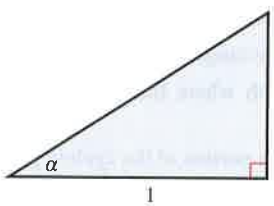


Figure for 12

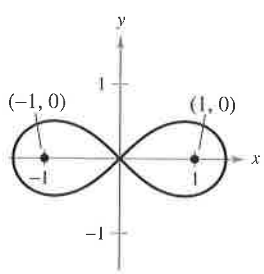
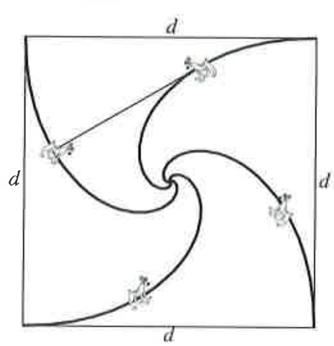


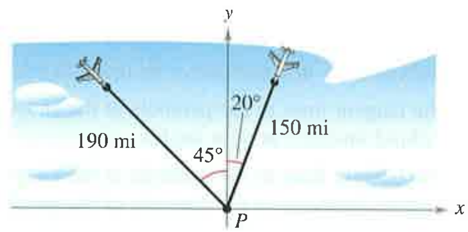
Figure for 13

13. Determine the polar equation of the set of all points (r, θ), the product of whose distances from the points $(1, 0)$ and $(-1, 0)$ is equal to 1, as shown in the figure.

14. Four dogs are located at the corners of a square with sides of length d. The dogs all move counterclockwise at the same speed directly toward the next dog, as shown in the figure. Find the polar equation of a dog's path as it spirals toward the center of the square.



15. An air traffic controller spots two planes at the same altitude flying toward each other (see figure). Their flight paths are 20° and 315°. One plane is 150 miles from point P with a speed of 375 miles per hour. The other is 190 miles from point P with a speed of 450 miles per hour.



(a) Find parametric equations for the path of each plane where t is the time in hours, with $t = 0$ corresponding to the time at which the air traffic controller spots the planes.

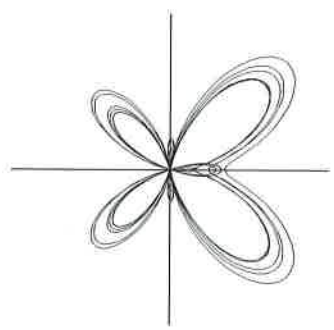
(b) Use the result of part (a) to write the distance between the planes as a function of t.

(c) Use a graphing utility to graph the function in part (b). When will the distance between the planes be minimum? If the planes must keep a separation of at least 3 miles, is the requirement met?

16. Use a graphing utility to graph the curve shown below. The curve is given by

$$r = e^{\cos \theta} - 2 \cos 4\theta + \sin^5 \frac{\theta}{12}.$$

Over what interval must θ vary to produce the curve?



FOR FURTHER INFORMATION For more information on this curve, see the article "A Study in Step Size" by Temple H. Fay in *Mathematics Magazine*. To view this article, go to the website *www.matharticles.com*.

17. Use a graphing utility to graph the polar equation $r = \cos 5\theta + n \cos \theta$, for $0 \leq \theta < \pi$ and for the integers $n = -5$ to $n = 5$. What values of n produce the "heart" portion of the curve? What values of n produce the "bell" portion? (This curve, created by Michael W. Chamberlin, appeared in *The College Mathematics Journal*.)