Particle Motion Solutions

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted.

Multiple Choice Solutions

1. E (2003 AB25)

$$x(t) = 2t^3 - 21t^2 + 72t - 3$$
$$v(t) = x'(t) = 6t^2 - 42t + 72 = 0$$
$$6\left(t^2 - 7t + 12\right) = 0$$
$$\left(t-3\right)(t-4) = 0 \Rightarrow t = 3, 4$$

2. A (2008 AB21/BC21)

V is increasing when $v'(t) > 0 \Rightarrow a(t) > 0$ which occurs when $x(t)$ is concave up, so $0 < t < 2$.

3. B (2008 AB7)

Using Fundamental Theorem of Calculus:

$$x(1) = x(0) + \int_0^1 (3t^2 + 6t)\,dt$$
$$x(1) = 2 + (t^3 + 3t^2)\Big|_{t=0}^{t=1}$$
$$x(1) = 2 + (4 - 0) = 6$$

Alternatively:

$$v(t) = 3t^2 + 6t$$
$$x(t) = t^3 + 3t^2 + c$$
$$x(0) = 0^3 + 6(0^2) + c = 2$$
$$c = 2$$
$$x(t) = t^3 + 3t^2 + 2$$
$$x(1) = 1 + 3 + 2 = 6$$

4. D (1985 AB14)

$v(t) > 0$ for all $t > 0$ therefore,

$$x(t) = \int_0^4 |v(t)|\,dt = \int_0^4 \left(3t^{\frac{1}{2}} + 5t^{\frac{3}{2}}\right)dt$$
$$= \left(2t^{\frac{3}{2}} + 2t^{\frac{5}{2}}\right)\Bigg|_{t=0}^{t=4}$$
$$= 16 + 64 = 80 \text{ meters}$$

5. C (1985 AB28)

   Average velocity of the particle is $\dfrac{\Delta s}{\Delta t} = \dfrac{-5(3)^2 + 5(0)}{3-0} = -15$.

6. B (1988 BC12 appropriate for AB)

   $v(t) = \int 3dt = 3t + C$ and $v(2) = 10$

   $10 = 3(2) + C$

   $4 = C$

   Distance traveled from $v(0) = 4$ and $v(2) = 10$

   $x(t) = \int_0^2 (3t+4)dt$

   $= \left(\dfrac{3}{2}t^2 + 4t\right)\Bigg|_{t=0}^{t=2}$

   $= 6 + 8 = 14$ meters

7. C (2008 AB86)

   $v(3) = x'(3) = 0$, so $x(t)$ has a horizontal tangent at $t = 3$; therefore, the only possible graphs are C and E. From the table, $v(1) = x'(1) = 2$, so $x(t)$ is increasing at $t = 1$, so the answer is C.

8. C (2003 AB76)

   Using the derivative function on the calculator:

   $v'(t) = a(t)$

   $a(4) = 1.633$

9. E (2003 AB91/BC91)

   Using the Fundamental Theorem of Calculus and the integral function on the calculator:

   $v(2) = v(1) + \int_1^2 \ln\left(1 + 2^t\right)\, dt$

   $v(2) = 2 + \int_1^2 \ln\left(1 + 2^t\right)\, dt = 3.346$

10. A (2003 AB83)

    Average velocity of a function on $[0, 3]$:

    $\dfrac{1}{3-0}\int_0^3 (e^t + te^t)\, dt = 20.086 \dfrac{\text{feet}}{\text{second}}$

Free Response
11. (2000 AB2/BC2)

(a) Runner A: velocity $=\frac{10}{3}\cdot 2=\frac{20}{3}$ $=6.666$ or $6.667\ \frac{\text{m}}{\text{sec}}$ Runner B : $v(2)=\frac{48}{7}=6.857\ \frac{\text{m}}{\text{sec}}$	2 { 1: velocity for Runner A 1: velocity for Runner B
(b) Runner A: acceleration $=\frac{10}{3}=3.333$ meters / sec^2 Runner B: $a(2)=v'(2)=\left.\frac{72}{(2t+3)^2}\right\|_{t=2}$ $=\frac{72}{49}=1.469$ meters / sec^2	2 { 1: acceleration for Runner A 1: acceleration for Runner B
(c) Runner A: distance $=\frac{1}{2}(3)(10)+7(10)=85$ meters Runner B: distance $=\int_0^{10}\frac{24t}{2t+3}\,dt=83.336$ meters	4 { 2: distance for Runner A 1: method 1: answer 2: distance for Runner B 1: integral 1: answer

12. (1999 AB1)

(a) $v(1.5) = 1.5\sin(1.5^2) = 1.167$ Up, because $v(1.5) > 0$	1: answer and reason
(b) $a(t) = v'(t) = \sin t^2 + 2t^2 \cos t^2$ $a(1.5) = v'(1.5) = -2.048 \text{ or } -2.049$ No, v is decreasing at 1.5 because $v'(1.5) < 0$	2 { 1: $a(1.5)$ 1: conclusion and reason
(c) $y(t) = \int v(t)\,dt$ $= \int t \sin t^2\,dt = -\frac{\cos t^2}{2} + C$ $y(0) = 3 = -\frac{1}{2} + C \Rightarrow C = \frac{7}{2}$ $y(t) = -\frac{1}{2}\cos t^2 + \frac{7}{2}$ $y(2) = -\frac{1}{2}\cos 4 + \frac{7}{2} = 3.826 \text{ or } 3.827$	3 { 1: $y(t) = \int v(t)\,dt$ 1: $y(t) = -\frac{1}{2}\cos t^2 + C$ 1: $y(2)$
(d) distance = $\int_0^2 \lvert v(t) \rvert\,dt = 1.173$ or $v(t) = t\sin t^2 = 0$ $t = 0 \text{ or } t = \sqrt{\pi} \approx 1.772$ $y(0) = 3;\ y\left(\sqrt{\pi}\right) = 4;\ y(2) = 3.826 \text{ or } 3.827$ $\left[y\left(\sqrt{\pi}\right) - y(0)\right] + \left[y\left(\sqrt{\pi}\right) - y(2)\right]$ $= 1.173 \text{ or } 1.174$	3 { 1: limits of 0 and 2 on an integral of $v(t)$ or $\lvert v(t) \rvert$ or uses $y(0)$ and $y(2)$ to compute distance 1: handles change of direction at student's turning point 1: answer 0/1 if incorrect turning point

13. (2005 Form B AB3)

(a) $a(4) = v'(4) = \frac{5}{7}$	1: answer
(b) $v(t) = 0$ $t^2 - 3t + 3 = 1$ $t^2 - 3t + 2 = 0$ $(t-2)(t-1) = 0$ $t = 1,\ 2$ $v(t) > 0$ for $0 < t < 1$ $v(t) < 0$ for $1 < t < 2$ $v(t) > 0$ for $2 < t < 5$	3 { 1: sets $v(t) = 0$ 1: direction change at $t = 1,\ 2$ 1: interval with reason
(c) $s(t) = s(0) + \int_0^t \ln(u^2 - 3u + 3)\,du$ $s(2) = 8 + \int_0^2 \ln(u^2 - 3u + 3)\,du$ $= 8.368$ or 8.369	3 { 1: $\int_0^2 \ln(u^2 - 3u + 3)\,du$ 1: handles initial condition 1: answer
(d) $\frac{1}{2}\int_0^2 \lvert v(t) \rvert\,dt = 0.370$ or 0.371	2 { 1: integral 1: answer

14. (2008 AB4/BC4)

(a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$. $x(3) = -2 + \int_0^3 v(t)\,dt = -2 - 8 = -10$ $x(6) = -2 + \int_0^6 v(t)\,dt = -2 - 8 + 3 - 2 = -9$ Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$	3: { 1: identifies $t = 3$ as a candidate 1: considers $\int_0^6 v(t)\,dt$ 1: conclusion }
(b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$. By the Intermediate Value Theorem, there are three values of t for which the particle is at $x(t) = -8$.	3: { 1: position at $t = 3$, $t = 5$, and $t = 6$ 1: description of motion 1: conclusion }
(c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and v is increasing.	1: answer with reason
(d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals.	2: { 1: answer 1: justification }