Derivatives and Their Applications Solutions

Multiple Choice:

1. A (1993 AB24) DL: 4

$$f(x)=\left(x^2-2x-1\right)^{\frac{2}{3}} \rightarrow f'(x)=\frac{2}{3}\left(x^2-2x-1\right)^{\frac{-1}{3}}(2x-2)=\frac{2(2x-2)}{3\left(x^2-2x-1\right)^{\frac{1}{3}}}$$

$$f'(0)=\frac{-4}{-3}=\frac{4}{3}$$

2. E (1985 AB2) DL: 4

$$f'(x)=4(2x+1)^3(2)=8(2x+1)^3$$

$$f''(x)=24(2x+1)^2(2)=48(2x+1)^2$$

$$f'''(x)=96(2x+1)^1(2)=192(2x+1)$$

$$f^{(4)}(x)=384$$

3. A (AP-like) DL: 4

$$h'(x)=f(x)g'(x)+g(x)f'(x)+\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

$$h'(5)=(3)(5)+(-2)(4)+\frac{(-2)(4)-(3)(5)}{(-2)^2}=7-\frac{23}{4}=\frac{5}{4}$$

4. A (AP-like) DL: 3

$$h'(x)=f'\left(g(2x)\right)g'(2x)(2)$$

$$h'(2)=f'\left(g(4)\right)g'(4)(2)=f'(3)g'(4)(2)=(-5)(9)(2)=-90$$

5. C (AP-like) DL: 3

$$h'(x)=2g(x)g'(x)$$

$$h'(3)=2g(3)g'(3)=2(6)(2)=24$$

6. E (1988 AB18) DL: 3

$$\frac{dy}{dx}=-2\sin\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)=-\sin\left(\frac{x}{2}\right); \quad \frac{d^2y}{dx^2}=-\cos\left(\frac{x}{2}\right)\left(\frac{1}{2}\right)=\frac{-1}{2}\cos\left(\frac{x}{2}\right)$$

7. D (AB Sample Question #7 from AP Calculus Course and Exam Description) DL: 4

8. B (2003 AB26) DL: 3

$3y^2 - 2x^2 + 2xy = 6$ (moving $-2xy$ from right to the left side of the equation will create a positive coefficient in front of the term that requires product rule and will avoid some potential errors)

$$6y\frac{dy}{dx} - 4x + 2x\frac{dy}{dx} + y(2) = 0$$

$$\frac{dy}{dx}(6y + 2x) = 4x - 2y$$

$$\frac{dy}{dx} = \frac{4x - 2y}{6y + 2x} = \frac{4(3) - 2(2)}{6(2) + 2(3)} = \frac{8}{18} = \frac{4}{9}$$

9. B (AB Sample Question #18 from AP Calculus Course and Exam Description) DL: 3

Instantaneous rate of change is another name for the derivative. Students should use their calculators to evaluate the derivative of the given function at $t = 90$ days.

10. D (2008 AB16) DL: 4

$$\cos(xy)\left[x\frac{dy}{dx} + y(1)\right] = 1$$

$$x\frac{dy}{dx} + y = \frac{1}{\cos(xy)} \rightarrow x\frac{dy}{dx} = \frac{1}{\cos(xy)} - y = \frac{1 - y\cos(xy)}{\cos(xy)} \rightarrow \frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy)}$$

11. A (1995 BC5, appropriate for AB) DL: 4

$$h'(x) = f'(g(x))g'(x)$$

$h''(x)$ requires the use of the product rule,

$$h''(x) = f'(g(x))g''(x) + g'(x)f''(g(x))g'(x)$$

$$h''(x) = f'(g(x))g''(x) + (g'(x))^2 f''(g(x))$$

12. A (AP-like) DL: 3

$$f'(x) = \frac{1}{1 + x^2}$$

$$f'(\sqrt{3}) = \frac{1}{1 + (\sqrt{3})^2} = \frac{1}{4}$$

13. C (AP-like) DL: 4

$$f(x)=\sqrt{e^x+3}\rightarrow f(0)=2$$

$$f'(x)=\frac{e^x}{2\sqrt{e^x+3}}\rightarrow f'(0)=\frac{1}{4}$$

Tangent line equation : $y=\frac{1}{4}(x-0)+2$, when $x=\frac{1}{2}$, $y=\frac{1}{8}+2=2.125$

14. A (AP-like) DL: 3

$$f'(x)=\frac{1-3e^{-3x}}{x+4+e^{-3x}} \quad \rightarrow \quad f'(0)=\frac{1-3e^0}{0+4+e^0}=\frac{-2}{5}$$

15. D (2008 AB3) DL: 4

$$f'(x)=(x-1)3\left(x^2+2\right)^2(2x)+\left(x^2+2\right)^3(1)$$

factoring out the common term $\left(x^2+2\right)^2$ helps to condense the answer;

$$f'(x)=\left(x^2+2\right)^2\left[(x-1)6x+\left(x^2+2\right)\right]$$

$$f'(x)=\left(x^2+2\right)^2\left[6x^2-6x+x^2+2\right]=\left(x^2+2\right)^2\left[7x^2-6x+2\right]$$

16. A (2008 AB28) DL: 4

If $f(x)$ contains (a,b) then its inverse $g(x)=f^{-1}(x)$ will contain (b,a). Their derivatives are related with the equation $g'(b)=\frac{1}{f'(a)}$.

Similarly, when $f(x)$ contains $(6,3)$ then its inverse $g(x)=f^{-1}(x)$ contains $(3,6)$.

Then $g'(3)=\frac{1}{f'(6)}=\frac{1}{-2}$.

17. D (AP-like) DL: 4

slope of the secant line ; $m = \frac{15-3}{4-1} = 4$

$h'(x) = 3x^2 + 2kx$

$h'(-2) = 12 - 4k$

setting $h'(-2) = 4$ and solving for k yields $k = 2$

18. D (AB Sample Question #5 from AP Calculus Course and Exam Description) DL: 4

The function $g(x)$ and its tangent line will contain the same point (point of tangency), therefore $g\left(\frac{1}{2}\right) = y\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right) + 1 = 3$

At the point of tangency, they also have the same slope, therefore $g'\left(\frac{1}{2}\right) = y'\left(\frac{1}{2}\right) = 4$

$g\left(\frac{1}{2}\right) + g'\left(\frac{1}{2}\right) = 4 + 3 = 7$

19. A (AB Sample Question #4 from the early version of AP Calculus Course and Exam Description) DL: 4

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \rightarrow \quad -2\pi = 4\pi(5)^2 \frac{dr}{dt} \quad \rightarrow \quad \frac{dr}{dt} = \frac{-2}{4(25)} = \frac{-1}{50}$

$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi(5)\frac{-1}{50} = \frac{-4\pi}{5}$

20 . B (AP-like) DL: 4

$f(x) = e^{\sin x} - 1 = 0 \quad \rightarrow \quad e^{\sin x} = 1 \quad \rightarrow \quad \sin(x) = \ln 1 = 0 \quad \rightarrow \quad x = \pi$ is the only solution on [1,4]

$f'(x) = e^{\sin x} \cos x$

$f'(\pi) = e^{\sin \pi} \cos \pi = e^0(-1) = -1$

21 . A (2008 AB18) DL: 5

The tangent line, $y = -x + k$, and the graph of the function y has the same slope, -1, at the point of tangency;

$y' = 2x + 3 = -1$ at $x = -2$

At the point of tangency, they also have the same y-coordinate;

$4 - 6 + 1 = 2 + k$

$k = -3$

22 . D (1998 AB90) DL: 4

$$A = \frac{1}{2}bh$$

$$\frac{dA}{dt} = \frac{1}{2}\left(b\frac{dh}{dt} + h\frac{db}{dt}\right)$$

$$\frac{dA}{dt} = \frac{1}{2}\left(b(-3) + h(3)\right) = \frac{3}{2}(-b + h)$$

When $b > h$, $\frac{dA}{dt}$ is negative therefore A decreases.

23. A (2008 AB11) DL: 3

$$f'(x) = x3(1-2x)^2(-2) + (1-2x)^3$$

$$f'(1) = -6 + (-1) = -7$$

24. B (AB Sample Question #3 from AP Calculus Course and Exam Description) DL: 4

$$f'(x) = \cos\left(\ln(2x)\right)\frac{1}{2x}(2)$$

$$f'(x) = \frac{\cos\left(\ln(2x)\right)}{x}$$

25. D (AP-like) DL: 4

$A = 2xy = 2x\left(9 - x^2\right) = 18x - 2x^3$

$\frac{dA}{dx} = 18 - 6x^2 = 0$ and changes sign from positive to negative at $x = \sqrt{3}$.

The domain is x : $(0,3)$, at $x = \sqrt{3}$, there must be the absolute max of the area function.

$A\big|_{x=\sqrt{3}} = 2\sqrt{3}\left(9 - \left(\sqrt{3}\right)^2\right) = 12\sqrt{3}$

26. B (AP-like) DL: 4

$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx} + \frac{dy}{dx}$

at $x = 3$ and $y = 1$, $\frac{dy}{dx} = 1^2 + 1 = 2$

then, $\frac{d^2y}{dx^2} = 2(1)(2) + (2) = 6$

Free Response

27. 2003-Form B- AB6a

(a) $f''(x) = \sqrt{f(x)} + x \cdot \frac{f'(x)}{2\sqrt{f(x)}} = \sqrt{f(x)} + \frac{x^2}{2}$

$f''(3) = \sqrt{25} + \frac{9}{2} = \frac{19}{2}$

$3 : \begin{cases} 2 : f''(x) \\ \quad < -2 > \text{ product or} \\ \quad \text{chain rule error} \\ 1 : \text{value at } x = 3 \end{cases}$

28. 2000 AB/BC 5

(a) $y^2 + 2xy\dfrac{dy}{dx} - 3x^2y - x^3\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}\left(2xy - x^3\right) = 3x^2y - y^2$

$\dfrac{dy}{dx} = \dfrac{3x^2y - y^2}{2xy - x^3}$

$2\begin{cases} 1: \text{implicit differentiation} \\ 1: \text{verifies expression for } \dfrac{dy}{dx} \end{cases}$

(b) When $x = 1$, $y^2 - y = 6$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = 3,\ y = -2$

At $(1,3)$, $\dfrac{dy}{dx} = \dfrac{9-9}{6-1} = 0$

Tangent line equation is $y = 3$

At $(1,-2)$, $\dfrac{dy}{dx} = \dfrac{-6-4}{-4-1} = \dfrac{-10}{-5} = 2$

Tangent line equation is $y + 2 = 2(x-1)$

$4\begin{cases} 1: \ y^2 - y = 6 \\ 1: \ \text{solves for } y \\ 2: \ \text{tangent lines} \end{cases}$

Note: 0/4 if not solving an equation of the form $y^2 - y = k$

(c) Tangent line is vertical when $2xy - x^3 = 0$

$x\left(2y - x^2\right) = 0$ gives $x = 0$ or $y = \dfrac{1}{2}x^2$

There is no point on the curve with x-coordinate 0.

When $y = \dfrac{1}{2}x^2$, $\dfrac{1}{4}x^5 - \dfrac{1}{2}x^5 = 6$

$-\dfrac{1}{4}x^5 = 6$

$x = \sqrt[5]{-24}$

$3\begin{cases} 1: \ \text{sets denominator of } \dfrac{dy}{dx} \text{ equal to } 0 \\ 1: \ \text{substitutes } y = \dfrac{1}{2}x^2 \text{ or } x = \pm\sqrt{2y} \\ \quad \text{into the equation for the curve} \\ 1: \ \text{solves for } x\text{-coordinate} \end{cases}$

29. 2014 AB/BC 1a,b

(a) $\dfrac{A(30)-A(0)}{30-0}=-0.197(\text{or } -0.196)\text{lbs/day}$	1: answer with units
(b) $A'(15)=-0.164(\text{or } -0.163)$ The amount of grass clippings in the bin is decreasing at a rate of 0.164 (or 0.163) lbs/day at time $t=15$ days.	2 { 1: $A'(15)$ 1: interpretation

30. 2012 AB/BC 1a

(a) $W'(12)\approx\dfrac{W(15)-W(9)}{15-9}=\dfrac{67.9-61.8}{6}$ $=1.017$ (or 1.016) The water temperature is increasing at a rate of approximately 1.017 °F per minute at time $t=12$ minutes.	2 : { 1 : estimate 1 : interpretation with units

31. 2001 AB/BC 2a

(a) Difference quotient; e.g. $W'(12)\approx\dfrac{W(15)-W(12)}{15-12}=-\dfrac{1}{3}$ °C/day or $W'(12)\approx\dfrac{W(12)-W(9)}{12-9}=-\dfrac{2}{3}$ °C/day or $W'(12)\approx\dfrac{W(15)-W(9)}{15-9}=-\dfrac{1}{2}$ °C/day	2 : { 1 : difference quotient 1 : answer (with units)

32. 2010 AB6a,b

(a) $f'(1) = \left.\dfrac{dy}{dx}\right\|_{(1,\,2)} = 8$ An equation of the tangent line is $y = 2 + 8(x-1)$.	2 { 1: $f'(1)$ 1: answer
(b) $f(1.1) \approx 2.8$ Since $y = f(x) > 0$ on the interval $1 \le x \le 1.1$, $\dfrac{d^2y}{dx^2} = y^3\left(1 + 3x^2y^2\right) > 0$ on this interval. Therefore on the interval $1 < x < 1.1$, the line tangent to the graph of $y = f(x)$ at $x = 1$ lies below the curve and the approximation 2.8 is less than $f(1.1)$.	2 { 1: approximation 1: conclusion with explanation

33. 2002-Form B-AB6

(a) Distance $= \sqrt{3^2 + 4^2} = 5$ km	1 : answer
(b) $r^2 = x^2 + y^2$ $2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$ or explicitly: $r = \sqrt{x^2 + y^2}$ $\frac{dr}{dt} = \frac{1}{2\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)$ At $x = 4$, $y = 3$, $\frac{dr}{dt} = \frac{4(-15) + 3(10)}{5} = -6$ km/hr	4 { 1 : expression for distance 2 : differentiation with respect to t $< -2 >$ chain rule error 1 : evaluation
(c) $\tan\theta = \frac{y}{x}$ $\sec^2\theta\,\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2}$ At $x = 4$ and $y = 3$, $\sec\theta = \frac{5}{4}$ $\frac{d\theta}{dt} = \frac{16}{25}\left(\frac{10(4) - (-15)(3)}{16}\right)$ $= \frac{85}{25} = \frac{17}{5}$ radians/hr	4 { 1 : expression for θ in terms of x and y 2 : differentiation with respect to t $< -2 >$ chain rule, quotient rule, or transcendental function error note: 0/2 if no trig or inverse trig function 1 : evaluation

34. 2014 AB/BC 3d

(d) $p'(x) = f'(x^2 - x)(2x - 1)$ $p'(-1) = f'(2)(-3) = (-2)(-3) = 6$	3 { 2: $p'(x)$ 1: answers

35. 2011-Form B- AB 2c

(c) $r'(3) = 50$ The rate at which water is draining out of the tank at time $t = 3$ hours is increasing at 50 liters/hour^2.	2 { 1: $r'(3)$ 1: meaning of $r'(3)$

36. 2009-FormB- AB5a

(a) $g(1) = e^{f(1)} = e^2$ $g'(x) = e^{f(x)} f'(x),\ \ g'(1) = e^{f(1)} f'(1) = -4e^2$ The tangent line is given by $y = e^2 - 4e^2(x-1)$.	$3 : \begin{cases} 1 : g'(x) \\ 1 : g(1) \text{ and } g'(1) \\ 1 : \text{tangent line equation} \end{cases}$

37. AP-like

$h'(0)$ does not exist.

Even if $\lim_{x\to 0^-} h'(x) = \lim_{x\to 0^-} 4x^3 + 3 = 3$ and $\lim_{x\to 0^+} h'(x) = \lim_{x\to 0^-} 3e^{3x} = 3$, since $\lim_{x\to 0^-} h(x) \neq \lim_{x\to 0^+} h(x)$, $h(x)$ is not continuous at $x = 0$, therefore $h(x)$ is not differentiable at $x = 0$.

General Rules

Definition of Derivative:

$$\frac{d}{dx}\left(f(x)\right) = f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Sum and Difference Rule:

$$\frac{d}{dx}\left(f(x)\pm g(x)\right) = f'(x)\pm g'(x)$$

Constant Multiple Rule:

$$\frac{d}{dx}\left(k\cdot f(x)\right) = k\cdot f'(x)$$

Product Rule:

$$\frac{d}{dx}\left(f(x)\cdot g(x)\right) = f'(x)g(x)+f(x)g'(x)$$

Quotient Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

Chain Rule:

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x))\cdot g'(x)$$

Particular Rules

Power Rule:

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Exponential functions:

$$\frac{d}{dx}\left(e^u\right) = e^u\cdot\frac{du}{dx}$$

Bases Other than e: $\frac{d}{dx}(a^u) = a^u(\ln u)\frac{du}{dx}$

Trigonometric Functions:

$$\frac{d}{dx}\left(\sin(u)\right) = \cos(u)\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left(\cos(u)\right) = -\sin(u)\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left(\tan(u)\right) = \sec^2(u)\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left(\csc(u)\right) = -\csc(u)\cot(u)\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left(\sec(u)\right) = \sec(u)\tan(u)\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left(\cot(u)\right) = -\csc^2(u)\cdot\frac{du}{dx}$$

Logarithmic Functions:

$$\frac{d}{dx}\left(\ln(u)\right) = \frac{1}{u}\cdot\frac{du}{dx}$$

Inverse Trigonometric Functions:

$$\frac{d}{dx}\left(\sin^{-1}(u)\right) = \frac{1}{\sqrt{1-u^2}}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}\left(\tan^{-1}(u)\right) = \frac{1}{1+u^2}\cdot\frac{du}{dx}$$