INTEGRATION USING DATA

Ex. Water is flowing into a tank over a 24-hour period. The rate at which water is flowing into the tank at various times is measured, and the results are given in the table below, where $R(t)$ is measured in gallons per hour and t is measured in hours. The tank contains 150 gallons of water when $t = 0$.

t (hours)	0	4	8	12	16	20	24
$R(t)$ (gal/hr)	8	8.8	9.3	9.2	8.9	8.1	6.7

(a) Estimate the number of gallons of water in the tank at the end of 24 hours by using a midpoint Riemann sum with three subintervals and values from the table. Show the computations that lead to your answer.

(b) Estimate the number of gallons of water in the tank at the end of 24 hours by using a trapezoidal sum with three subintervals and values from the table. Show the computations that lead to your answer.

(c) A model for this function is given by $W(t) = \frac{1}{75}\left(600 + 20t - t^2\right)$. Use the model to find the number of gallons of water in the tank at the end of 24 hours.

(d) Use the model given in (c) to find the average rate of water flow over the 24-hour period.

Homework: Worksheet

CALCULUS

EXPLORATION OF THE SECOND FUNDAMENTAL THEOREM OF CALCULUS

$$\frac{d}{dx}\int_1^x t^2\,dt =$$

$$\frac{d}{dx}\int_{\pi/6}^x \cos t\,dt =$$

Second Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dt =$$

$$\frac{d}{dx}\int_x^4 t^2\,dt =$$

$$\frac{d}{dx}\int_x^a f(t)\,dt =$$

$$\frac{d}{dx}\int_{\pi/6}^{x^2} \cos t\,dt =$$

Second Fundamental Theorem of Calculus (Chain Rule Version):

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt =$$

Ex. Use the Second Fundamental Theorem to evaluate:

(a) $\frac{d}{dx}\int_3^x \sqrt{1+t^2}\,dt =$

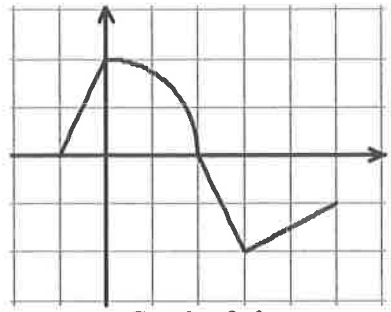
(b) $\frac{d}{dx}\int_2^x \tan\left(t^3\right)dt =$

(c) $\frac{d}{dx}\int_{-1}^{x^3} \frac{1}{1+t}\,dt =$

(d) $\frac{d}{dx}\int_2^{\sin x} \sqrt[3]{1+t^2}\,dt =$

Ex. The graph of a function f consists of a quarter circle and line segments. Let g be the function given by

$$g(x) = \int_0^x f(t)\,dt.$$



Graph of f

(a) Find $g(0), g(-1), g(2), g(5)$.

(b) Find all values of x on the open interval $(-1, 5)$ at which g has a relative maximum. Justify your answer.

(c) Find the absolute minimum value of g on $[-1, 5]$ and the value of x at which it occurs. Justify your answer.

(d) Find the x-coordinate of each point of inflection of the graph of g on $(-1, 5)$. Justify your answer.

Homework: Worksheet

5.1 The Natural Logarithmic Function and Differentiation

Definition of the Natural Logarithmic Function:

The natural logarithmic function is defined by $\ln x = \int_1^x \frac{1}{t}\,dt$, where $x > 0$.

The base of natural logs is the number e. e was named for a Swiss mathematician, Leonhard Euler.

By definition: $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n \approx 2.7183...$

$y = \ln x$ and $y = e^x$ are inverses.

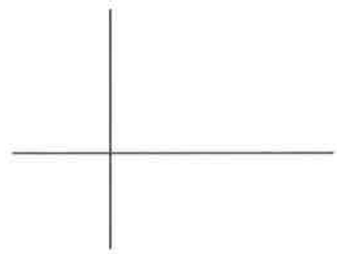
Properties of Natural Logs:

1) Domain of $y = \ln x$ is $(0, \infty)$.

Range of $y = \ln x$ is $(-\infty, \infty)$.

2) The graph of $y = \ln x$ is continuous, increasing, and one-to-one.

3) The graph of $y = \ln x$ is concave downward.



Other properties:

If a and b are positive numbers and n is rational, then:

1) $\ln 1 = 0$

2) $\ln e = 1$

3) $\ln ab = \ln a + \ln b$

4) $\ln \frac{a}{b} = \ln a - \ln b$

5) $\ln a^n = n \ln a$

Ex. Write as a sum, difference, or multiple of logs:

$$\ln \frac{\left(x^2+3\right)^2}{\sqrt[3]{x^2+1}} =$$

Ex. Write as a single log:

$$2\ln(x+3) + \frac{1}{2}\ln(x-2) =$$

$$\boxed{\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}, \quad u > 0}$$

Ex. $y = \ln(2x)$

$y' =$

$$\boxed{\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}, \quad u > 0}$$

Ex. $f(x) = \ln(x^2 + 1)$

$f'(x) =$

Ex. $y = x \ln x$

$y' =$

Ex. $f(x) = \ln \sqrt{x+1}$. Find $f'(x)$.

Ex. $y = \ln(\ln x)$

$y' =$

Ex. $y = \ln(x^3)$

$y' =$

Ex. $y = (\ln x)^3$

$y' =$

Ex. Show that $y = x \ln x - 4x$ is a solution to the differential equation

$x + y - xy' = 0$.

Homework:

P. 329: 15, 19, 23, 25, 30, 45, 49, 51, 55 – 63 odd, 75, 77, 81

5.2 The Natural Log Function and Integration

In the previous section we learned that $\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}$.

The corresponding integration formula is:

$$\boxed{\int \frac{1}{u}\,du = \ln|u| + C}$$

Ex. $\int \frac{2}{x}\,dx =$

Ex. $\int_1^e \frac{2}{x}\,dx =$

Ex. $\int \frac{1}{2x-1}\,dx =$

Ex. $\int \frac{3x^2+1}{x^3+x}\,dx =$

Ex. $\int_1^e \frac{(1+\ln x)^3}{x}\,dx =$

Ex. $\int_e^{e^2} \frac{(\ln x)^4}{x}\,dx =$

Ex. $\int_0^3 \frac{x^2-5}{x+2}\,dx =$

***If you are integrating a quotient and the power of the numerator is greater than or equal to the power of the denominator, you must ______________________________.

Four more integration formulas to learn:

$$\int \tan u\,du = -\ln|\cos u| + C$$
$$\int \cot u\,du = \ln|\sin u| + C$$
$$\int \sec u\,du = \ln|\sec u + \tan u| + C$$
$$\int \csc u\,du = -\ln|\csc u + \cot u| + C$$

Why are these true?

$\int \tan x\,dx =$

$\int \sec x\,dx =$

Ex. $\int \tan(3x)\,dx =$

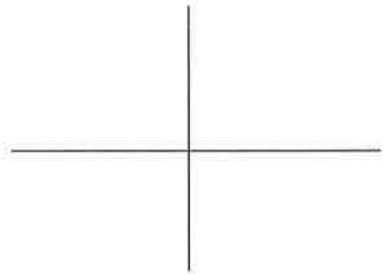
Homework: Page 338: 3, 7, 13, 17, 21, 25, 33, 37, 47, 53, 67

5.3 Inverse Functions

A function g is the inverse of a function f if and only if

$f(g(x)) = x$ for each x in the domain of g and

$g(f(x)) = x$ for each x in the domain of f.



The inverse of f is denoted f^{-1}.

Properties of inverses:

1) If g is the inverse of f, then f is the inverse of g.
2) The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
3) Not every function has an inverse, but if a function does have an inverse, the inverse is unique.

Ex. Given $f(x) = \sqrt{x-1}$.

(a) Find the inverse function of f.

(b) Graph f and f^{-1} on the same set of coordinate axes.

(c) Describe the relationship between the graphs.

(d) State the domain and range of f and f^{-1}.

Ex. Given $f(x) = x^3$ and $f^{-1}(x) = \sqrt[3]{x}$.

(a) $f(2) =$

(b) $f'(x) =$

(c) $f'(2) =$

(d) $f^{-1}(8) =$

(e) What is the derivative of $f^{-1}(x)$? It is called $(f^{-1})'(x)$.

$(f^{-1})'(x) =$

(f) $(f^{-1})'(8) =$

What do you notice about your answers to (c) and (f)?

Theorem 5-9 The Derivative of an Inverse Function
Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$ and $g'(x) = \dfrac{1}{f'(g(x))}$ so that

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$

Sometimes we are given all of the pieces we need, and then we just have to put them together to get the answer.

Ex. If $f(3) = 5$ and $f'(3) = \dfrac{7}{2}$, find $\left(f^{-1}\right)'(5)$.

When we're not given all of the pieces, we need to find $f^{-1}(a)$ and $f'(x)$ before we can use the formula.

Ex. Let $f(x) = x^3 + 2x - 1$. Find $\left(f^{-1}\right)'(2)$.

Ex. Let $g(x) = \sqrt{x+1}$. Find $\left(g^{-1}\right)'(2)$.

Ex. Let $f(x) = \cos x$, $0 \le x \le \pi$. Find $\left(f^{-1}\right)'\left(\dfrac{\sqrt{3}}{2}\right)$.

Homework: Worksheet

5.4 Exponential Functions

You learned in Precalculus:

$y = \log_b x$ means $x = b^y$ where $b > 0$ and $x > 0$

$y = \ln x$ means $x = e^y$ where $x > 0$

Ex. Solve. Give decimal answers correct to three decimal places.

(a) $e^{x+1} = 7$

(b) $\ln(2x-3) = 5$

Derivative of an exponential function: $\frac{d}{dx}\left[e^u\right] = e^u \frac{du}{dx}$

Ex. Find the derivative.

(a) $y = e^{3x^2}$

(b) $y = \sin^2\left(e^x\right)$

(c) $y = \ln\left(4 + e^{3x}\right)$

(d) $y = \ln\left(e^{x^3}\right)$

(e) $f(x) = \ln\left(\frac{3+e^x}{3-e^x}\right)$

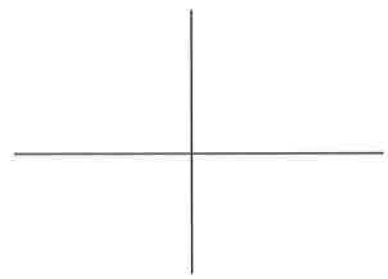
(f) $y = x^2 e^{-x}$

Ex. Use implicit differentiation to find $\dfrac{dy}{dx}$.

$$e^{xy} + x^2 - y^2 = 10$$

Ex. Find the relative extrema and the points of inflection, and sketch the graph, given

$$f(x) = xe^x$$



Homework:
P. 356: : 1,3,11, 35 – 49 odd

5.4 Exponential Functions –Day 2

Integral of an exponential function: $\int e^u\,du = e^u + C$

Ex. $\int e^{3x+1}dx =$

Ex. $\int 5xe^{-x^2}\,dx =$

Ex. $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx =$

Ex. $\int \sin x\, e^{\cos x}dx =$

Ex. $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx =$

Ex. $\int_{-1}^{0} e^x \cos\left(e^x\right)dx =$

Ex. Solve the differential equation

$$\frac{dy}{dx} = \left(e^{x} - e^{-x}\right)^{2}$$

Ex. Find the particular solution of the differential equation that satisfies the initial conditions.

$$f''(x) = \sin x + e^{2x}, \quad f(0) = \frac{1}{4}, \quad f'(0) = \frac{1}{2}$$

Ex. The rate at which water is being pumped into a tank is $r(t) = 20e^{0.02t}$, where t is in minutes and $r(t)$ is in gallons per minute. How many gallons of water have been pumped into the tank in the first five minutes? Use your calculator, and give your answer correct to three decimal places.

Homework:
P. 358: 85-105 odd