2012 Mock Exam Multiple Choice Correct Rationales-BC

1. E. Student writes $y=(\sin x)^3$ and uses the chain rule to determine $\frac{dy}{dx}=3\sin^2 x\cos x$.

2. A. Student determines that the particle is at rest when both $x'(t)$ and $y'(t)$ are equal to 0. $x'(t)=3t^2-6t=0;\ 3t(t-2)=0$ when $t=0,2.$ $y'(t)=12-6t=0$ when $t=2.$ Therefore the particle is at rest when $t=2.$ Since $x(2)=-4,$ $y(2)=12,$ the point is $(-4,\ 12)$.

3. B. Student sees the figure as a trapezoid and a rectangle, remembers that $\int_2^4 f(x)dx$ must be negative, and determines the total accumulation of area from $x=0$ to $x=4$ is equal to 0. $\int_0^2 f(x)\ dx+\int_2^4 f(x)\ dx=\frac{1}{2}(2)(1+5)+2(-3)=6-6=0$

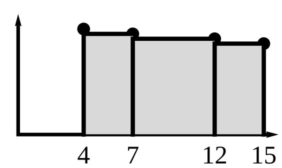
4. A. Student uses the arclength formula $\int_a^b \sqrt{1+(f'(x))^2}\,dx$. Since $y'=\frac{1}{x}$, the integral which gives the length of $y=\ln x$ from $x=1$ to $x=2$ is $\int_1^2 \sqrt{1+\frac{1}{x^2}}\ dx$.

5. C. Student determines the series is geometric with $r=-\frac{3}{4}$. The sum of the series is given by $\frac{a_1}{1-r}$; $f(3)=\frac{1}{1+\frac{3}{4}}=\frac{1}{\frac{7}{4}}=\frac{4}{7}$.

6. C. Student determines $du=2x\ dx;\ \frac{1}{2}du=x\ dx;\ u=4^2-3=13$ when $x=4$ and $u=(-1)^2-3=-2$ when $x=-1;$ $\int_{-1}^4 x(x^2-3)^5\ dx\ =\ \frac{1}{2}\int_{-2}^{13} u^5 du$

7. A. Student determines the derivative using implicit differentiation:
$\frac{1}{\sqrt{1-x^2}}=\frac{1}{y}\frac{dy}{dx};\ \frac{y}{\sqrt{1-x^2}}=\frac{dy}{dx}$

8. C. Student sketches the right-hand rectangles then adds
$50+3(6.2)+5(5.9)+3(5.6)=114.9$ liters
(Forgetting the original volume of 50 liters is a typical error.)



9. D. Student determines the series in I converges by the Ratio Test, II diverges by a Direct Comparison test, and III converges by a Direct Comparison Test.

I. $\lim\limits_{n\to\infty}\left[\dfrac{8^{n+1}}{(n+1)!}\dfrac{n!}{8^n}\right]=\lim\limits_{n\to\infty}\left[\dfrac{8}{(n+1)}\right]=0<1$

I. $n^n > n^{100}$ for $n>100$; $0<\dfrac{n!}{n^n}<\dfrac{n!}{n^{100}}$; Since $\sum\limits_{n=1}^{\infty}\dfrac{n!}{n^2}$ diverges, $\sum\limits_{n=1}^{\infty}\dfrac{n^1}{n^{100}}$ by the direct comparison test.

II. $0<\dfrac{n+1}{(n)(n+2)(n+3)}<\dfrac{1}{n^2}$. Since $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ is a converging p-series, $p>1$, the series converges by the direct comparison test.

10. E. Student evaluates $\int_1^4 t^{\frac{-3}{2}}\,dt=-2t^{\frac{-1}{2}}\Big|_{t=1}^{t=4}=-2\left(\dfrac{1}{\sqrt{4}}\right)-(-2)=-1+2=1$

11. A. Student determines f is continuous but not differentiable at $x=2$.

12. C. At $(-1,-1)$, $\dfrac{dy}{dx}=0$, $\dfrac{d^2y}{dx^2}=2x+(1)\dfrac{dy}{dx}=-2.$ Therefore the graph of f is concave down and $(-1,-1)$ is a local maximum of f .

13. C. Student uses the Ratio Test :
$\lim\limits_{n\to\infty}\left|\dfrac{(x-4)^{2(n+1)}}{3^{n+1}}\bullet\dfrac{3^n}{(x-4)^{2n}}\right|=\lim\limits_{n\to\infty}\left|\dfrac{(x-4)^2}{3}\right|<1\Rightarrow(x-4)^2<3$. Therefore the radius of convergence is $\sqrt{3}$.

14. E. Student recognizes the logistic growth differential equation model, $\dfrac{dy}{dt}=ky(1-y)$

15. A. Student determines that $h(6)<0,\ h'(6)=0,\ h''(6)>0$.

$h(6)=\int_0^6 f(t)\,dt<0$ since the accumulated area is below the $x-$axis;
$h'(x)=f(x)$ so $h'(6)=f(6)=0$
$h''(x)=f'(x)$ and f is increasing at $x=6$ therefore $h''(x)>0$

16. C. Student uses Euler's method with two intervals. $\Delta x=\dfrac{1}{2}$

$\dfrac{dy}{dx}_{(1,3)}=-2$ and $f\left(1\dfrac{1}{2}\right)\doteq f(1)+f'(1)\left(\dfrac{1}{2}\right)=3+-2\left(\dfrac{1}{2}\right)=2$.

$\dfrac{dy}{dx}_{(1.5,2)}=-\dfrac{1}{2}$ and $f(2)\doteq f\left(1\dfrac{1}{2}\right)+f'\left(1\dfrac{1}{2}\right)\cdot\left(\dfrac{1}{2}\right)=2+-\dfrac{1}{2}\cdot\dfrac{1}{2}=1\dfrac{3}{4}$

17. C. Student recognizes the given power series represents the series for $\sin x$ divided by x: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... + \frac{(-1)^n x^{2n+1}}{(2n+1)} + \ldots$;

$$\frac{\sin x}{x} = \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + ... + (-1)^n \frac{x^{2n}}{(2n+1)!} + ...$$

18. A. Student uses the Fundamental Theorem of Calculus to determine $\int_{-5}^{2} f'(x)\, dx = f(2) - f(-5)$; $f(-5) = f(2) - \int_{-5}^{2} f'(x)\, dx$; $1 - (3 - 2\pi) = 2\pi - 2$.

19 C. Student uses the quotient rule to determine $f\,'(x) = \frac{(x+2) - x}{(x+2)^2}$;

$\frac{2}{(x+2)^2} = \frac{1}{2}$; $4 = (x+2)^2$; $\pm 2 = x + 2$; $x = -4, 0$. Therefore the points are $(0,\ 0),\ (-4,\ 2)$.

20. C. Student uses partial fractions to rewrite

$$\frac{5x+8}{(x+2)(x+1)} = \frac{\text{A}}{x+2} + \frac{\text{B}}{x+1};$$

when $x = -2$, $\frac{5(-2)+8}{-2+1} = \text{A}$ so $\text{A} = 2$,

and when $x = -1$, $\frac{5(-1)+8}{-1+2} = \text{B}$ so $\text{B} = 3$.

$$\int_0^1 \left(\frac{2}{x+2} + \frac{3}{x+1} \right) dx = 2\ln|x+2| + 3\ln|x+1| \Big\|_{x=0}^{x=1} = 2\ln 3 + 3\ln 2 - 2\ln 2 = \ln \frac{9(8)}{4} = \ln(18).$$

21. E. Student recognizes that if the horizontal asymptote is $y = 5$, then $\lim_{x\to\infty} y = 5$. Since the ratio of the leading coefficients in E is $\frac{20}{4} = 5$, E is the correct choice.

22 D. Student determines that since the series converges at $x = 5$ and 5 is 2 units greater than 3, the radius of convergence must be at least 2. If 5 is an endpoint of the interval of convergence then the interval must be at least $1 < x \le 5$. The value 2 is within the interval; therefore, the series will definitely converge at 2.

23. A. Student recognizes the only answer choice which has an antiderivative which is linear.

24. E. Student uses integration by parts: $\int x^4 \sin x\ dx$

$u = x^4$; $du = 4x^3 dx$; $dv = \sin x$; $v = -\cos x$. Therefore $\int x^4 \sin x\ dx =$

$-x^4 \cos x + \int 4x^3 \cos x\ dx$

25. B. Student recognizes the improper integral and evaluates $\lim_{b\to\infty} \int_1^b xe^{-x^2} dx \Rightarrow$

$$\lim_{b\to\infty} -\frac{1}{2} e^{-x^2} \Bigg|_{x=1}^{x=b} = \lim_{b\to\infty} \left(-\frac{1}{2e^{b^2}} + \frac{1}{2e} \right) = \frac{1}{2e}$$

26. B. Student uses $x = (1 + 2\sin\theta)\cos\theta$ and $y = (1 + 2\sin\theta)\sin\theta$ and the product rule to determine $\dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \left.\dfrac{(1+2\sin\theta)(\cos\theta) + 2\cos\theta\sin\theta}{-(1+2\sin\theta)(\sin\theta) + 2\cos^2\theta}\right|_{\theta=0} = \dfrac{1}{2}$

27. C. Student recognizes the first series $\sum_{n=1}^{\infty} \frac{1}{n^{2p}}$ as a p-series so $2p > 1,\ p > \frac{1}{2}$ for convergence. The second series $\sum_{n=1}^{\infty} \left(\frac{p}{2}\right)^n$ is geometric so $\frac{p}{2} < 1;\ p < 2$ for convergence. Therefore the interval of convergence for both series is $\frac{1}{2} < p < 2$.

28. D. Student evaluates the limit by substitution and recognizes the indeterminate form $\frac{0}{0}$ so applies L'Hopital's rule: $\lim\limits_{x \to 1} \frac{g(x)}{g'(x)} = \frac{g(1)}{g'(1)} = \frac{6}{3} = 2.$

Calculator Allowed

76. B. Student recognizes that the derivative from the left of 0 and the derivative from the right of 0 are not equivalent, therefore f is not differentiable at 0.

77. B. Student uses the calculator to determine the intervals where the graph of $f'(x) - g'(x) > 0$. (This is more efficient than calculating the points of intersection between the two derivatives.)

78. C. Student uses geometry and the Fundamental Theorem of Calculus to calculate the accumulated area:

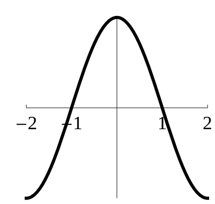
$$3\int_{-1}^{9} f(x)\,dx + \int_{-1}^{9} 2\,dx = 3(3(-1) + \frac{1}{2}(-1)(1) + \frac{1}{2}(6)(2)) + 2x\Big|_{-1}^{9} = 3\left(\frac{5}{2}\right) + (18+2) = \frac{55}{2} = 27.5$$

79. E. Student determines the third-degree Taylor polynomial for f about $x = 3$ is $f(3) + f'(3)(x-3) + f''(3)\frac{(x-3)^2}{2!} + f'''(3)\frac{(x-3)^3}{3!} =$

$$2 - \frac{(x-3)}{1!} + \frac{6(x-3)^2}{2!} + \frac{12(x-3)^3}{3!} = 2 - (x-3) + 3(x-3)^2 + 2(x-3)^3.$$

80. E. Student determines since f' changes sign from negative to positive at $x = -3$, f has a relative minimum at $x = -3$. Since $f'(x)$ does not change from increasing to decreasing or vice versa at $x = -2$, there is no point of inflection at $x = -2$. Since f' is decreasing on $0 < x < 4$, the graph is concave down on this interval

81. D. Student recognizes that the function is even since $f(x) = f(-x)$, $f''(x) > 0$ on $(-2,-1)$ and $(1,2)$; $f''(x) < 0$ on $(-1, 0)$ and $(0,1)$. The $x-$coordinates of the points of inflection are $x = -1$ and $x = 1$ since concavity changes sign at those values. Function looks similar to the one at the right.



82. C. Student calculates the average value by integrating

$$\left(\frac{1}{\frac{\pi}{2} - 0}\right)\int_{0}^{\frac{\pi}{2}} \sqrt{\cos x}\,dx = 0.763$$

83. C. Student recognizes the answer choice which illustrates the definition of continuity at a point: $f(3) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$.

84. D. Student identifies the intervals on which $f'(x)$ is decreasing and the graph of f is concave down.

$$f''(x) = \left(e^{x^4-2x^2+1}\right)\left(4x^3-4x\right)$$

$$f''(x) = \left(e^{x^4-2x^2+1}\right)(4x)\left(x-1(x+1\right)$$

$f''(x)$: $-$ on $(1.5, -1)$, $+$ on $(-1, 0)$, $-$ on $(0, 1)$, $+$ on $(1, 1.5)$

1.5		−1		0		1		1.5
	−		+		−		+	

85. B. Student calculates $F'(s) = \dfrac{dF}{ds} \bullet \dfrac{ds}{dt}$; $F'(50) = \dfrac{d}{ds}\left(6e^{\frac{s}{20}-\frac{s^2}{2400}}\right)\Bigg|_{s=50} \bullet 20 = 4.299$ mpg/hour.

86. B. Student selects the table of values where $f'(x) > 0$ so $f(x)$ is increasing and $\int_4^7 f(t) = 0$ implies that $f(x)$ consists of positive and negative values.

87. B. Student calculates $V = \int_0^2 (\ln(3-x))^2 \; dx = 1.029$.

88 E. Student determines when $x < 0$, f' is increasing so the graph of f is concave up; similarly, when $x > 0$, f' is decreasing so the graph of f is concave down.

89. E. Student uses the Fundamental Theorem of Calculus: $\int_0^3 \dfrac{t+3}{\sqrt{t^3+1}}\, dt = v(3) - v(0)$;

$$v(3) = 5 + \int_0^3 \frac{t+3}{\sqrt{t^3+1}}\, dt = 11.710$$

90. E. Student uses the comparison test. It is given that $\sum a(n)$ converges and since $n > 1$ for all n in the series, $\dfrac{a(n)}{n} < a(n)$. Therefore by the comparison test, $\sum a(n)$ must also converge.

91. D. Student calculates the area of the shaded regions by subtracting the area of the rose from the area of the circle: $\int_0^{2\pi} \left(\dfrac{1}{2} \bullet 2^2\right) d\theta - 3\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left(\dfrac{1}{2}(2\cos 3\theta)^2\right) d\theta = 9.425$.

92. B. Student determines that if $h(x)$ is defined to be below the $x-$axis it cannot have a positive definite integral value so I is false. Since $h(x) = h(2-x)$, $h'(x) = -h'(2-x)$; $h'(1) = -h'(2-1)$; $h'(1) = -h'(1)$; $2h'(1) = 0$; $h'(1) = 0$, II is true. Using the derivative argument for II, $h'(0) = -h'(2-0)$; $h'(0) = -h'(2)$, therefore $h'(0)$ cannot equal $h'(2)$ unless they are both 0, which contradicts the statement that they both equal 1, therefore III is also false.