NATIONAL MATH + SCIENCE INITIATIVE

AP Calculus

Taylor Series

Presenter Notes

2016-2017 EDITION

Student Study Session - Presenter Notes

Thank you for agreeing to present at one of NMSI's Saturday Study Sessions. We are grateful you are sharing your time and expertise with our students. Saturday mornings can be a "tough sell" for students, so we encourage you to incorporate strategies and techniques to encourage student movement and engagement. Suggestions for different presentation options are included in this document. If you have any questions about the content or about presentation strategies, please contact Mathematics Director Charla Holzbog at cholzbog@nms.org or AP Calculus Content Specialist Karen Miksch at kmiksch@nms.org.

The material provided contains many released AP multiple choice and free response questions as well as some AP-like questions that we have created. The goal for the session is to let the students experience a variety of both types of questions to gain insight on how the topic will be presented on the AP exam. It is also beneficial for the students to hear a voice other than their teacher in order to help clarify their understanding of the concepts.

Suggestions for presenting:
The vast majority of the study sessions are on Saturday and students and teachers are coming to be WOWed! We want activities to engage the students as well as prepare them for the AP Exam. The following presenter notes include pacing suggestions (you only have 50 minutes!), solutions, and recommended engagement strategies.

Suggestions on how to prepare:
- The notes/summaries on the last page(s) are for reference. We want the students' time during the session to be focused on the questions as much as possible and not taking or reading the notes. As the questions are presented during the session, you may wish to refer the students back to those pages as needed. It is not our intent for the sessions to begin with a lecture over these pages.
- As you prepare, work through the questions in the packet noting the level of difficulty and topic or skill required for the questions.
- Design a plan for what questions you would like to cover with the group depending on their level of expertise. Some groups will be ready for the tougher questions while other groups will need more guidance and practice on the easier ones. Create an easy, medium, and hard listing of the questions prior to the session. This will allow you to adjust on the fly as you get to know the groups. In most instances, there will **not** be enough time to cover all the questions in the packet. Use your judgement on the amount of questions to cover based on the students' interactions. Remember to include both multiple choice and free response type questions. Discussions on test taking strategies and scoring of the free response questions are always great to include during the day.
- The concepts should have been previously taught; however, be prepared to "teach" the topic if you find out the students have not covered the concept prior in class. In sessions where multiple schools come together, you might have a mixture of students with and without prior knowledge on the topic. You will have to use your best judgement in this situation.
- Consider working through some free response questions before the multiple choice questions, or flipping back and forth between the two types of questions. Sometimes, if free response questions are saved for the last part of the session, it is possible students only get practice with one or two of them and most students need additional practice with free response questions.

Taylor Series Student Session-Presenter Notes

This session includes a reference sheet at the back of the packet. We suggest that the presenter does not spend time going over the whole reference sheet, but may point it out to students that it is available to refer to if needed.

We suggest that students will work in pairs (depending on the size of the class), arrange the desks prior to the start of the session.

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use our presenter notes and your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted. Notice in the solutions guide the questions are categorized as 3, 4, or 5 indicating a typical question of the difficulty level (DL) for a student earning these qualifying scores on the AP exam.

I. 10 Minute Introductory Activity

- Arrange the students randomly into pairs. Ask students to work together on questions 1, 3, and 6 while you actively walk around to monitor their progress and assist with any questions. Point out the reference sheet in the back of the packet if needed for assistance. Use this time to gauge the level of knowledge of the group.
- After a few minutes check answers with students and clarify any misunderstandings.

II. 20 minutes Multiple Choice Additional Practice-*your choice of questions depending on student struggles during the first ten minutes—some suggested questions are listed below.*

- Model #12, 10, 5, 9 (notice the multiplication required in #5 and 9-this method has shown up in recent exams) asking for student input throughout, then have students work in pairs (think, pair, share) to complete #2, 4, 11, 7, 8, and 13 as time allows.

III. 20 minutes Free Response

- Model how to do the free response question involving a MacLaurin series by working through #17 with the students. Draw attention especially to part b where students are asked to express f' as a rational function, and also the multiplication required to do part c.
- As time allows, have students try 16c, #15, #14. If time is short, share the rubrics with students for these questions and focus on error, testing endpoints for intervals of convergence, how limits show up in series questions, and the operation on known series of term by term integration.
- Encourage students to do the remainder of the packet not covered during in the session with their teacher in their classes.

Taylor Series Solutions:
Multiple Choice:

1. B (BC Sample 5 from the early version of AP Calculus Course and Exam Description) DL: 4

$$T_3(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$T_3(x) = 2 + \frac{(-3)}{1!}(x-1) + \frac{(3)}{2!}(x-1)^2 + \frac{(-2)}{3!}(x-1)^3 = 2 - 3(x-1) + \frac{(3)}{2!}(x-1)^2 - \frac{1}{3}(x-1)^3$$

2. C (AP Calculus Course and Exam Description, BC Sample 8) DL:3

$$f(0) = 3$$

$$\frac{f'(0)}{1!} = -4 \rightarrow f'(0) = -4$$

$$\frac{f''(0)}{2!} = 2 \rightarrow f''(0) = 4$$

$$\frac{f'''(0)}{3!} = -3 \rightarrow f'''(0) = (-3)(3!) = -18$$

3. C (2012 BC13) DL: 4

$$a_n = \frac{(x-4)^{2n}}{3^n} \quad \text{and} \quad a_{n+1} = \frac{(x-4)^{2(n+1)}}{3^{n+1}} = \frac{(x-4)^{2n+2}}{3^{n+1}}$$

$$\lim_{n\to\infty}\left|\frac{\frac{(x-4)^{2n+2}}{3^{n+1}}}{\frac{(x-4)^{2n}}{3^n}}\right| = \lim_{n\to\infty}\left|\frac{(x-4)^{2n+2}}{3^{n+1}} \cdot \frac{3^n}{(x-4)^{2n}}\right| = \lim_{n\to\infty}\left|\frac{(x-4)^2}{3}\right| < 1$$

$$(x-4)^2 \cdot \lim_{n\to\infty}\left|\frac{1}{3}\right| < 1 \rightarrow (x-4)^2 \cdot \frac{1}{3} < 1 \rightarrow (x-4)^2 < 3 \rightarrow (x-4) < \sqrt{3}$$

4. C (2012 BC17) DL: 4

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

Therefore, $\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \cdots (-1)^n \frac{x^{2n}}{(2n+1)!} + \cdots$

5. C (AP-Like) DL:5

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\cdots$$

$$\cos^2 x = (\cos x)(\cos x) = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\cdots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots\cdot$$

$$- \frac{x^2}{2!} + \frac{x^4}{2!2!} - \frac{x^6}{2!4!} + \frac{x^8}{2!6!} \cdots\cdot$$

$$+ \frac{x^4}{4!} - \frac{x^6}{4!2!} + \frac{x^8}{4!4!} - \frac{x^{10}}{4!6!} \cdots\cdot$$

$$- \frac{x^6}{6!} + \frac{x^8}{6!2!} - \frac{x^{10}}{6!4!} + \frac{x^{12}}{6!6!} \cdots\cdot$$

adding x^4 terms gives $\frac{x^4}{4!} + \frac{x^4}{2!2!} + \frac{x^4}{4!} = \frac{x^4}{24} + \frac{x^4}{4} + \frac{x^4}{24} = \frac{x^4 + 6x^4 + x^4}{24} = \frac{8x^4}{24} = \frac{1}{3}x^4$

6. B (AP-Like) DL: 5
When the fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate f on the interval $[0,1]$,

$Error < \left|\frac{f^{(5)}(z)}{(5)!}(x-0)^5\right|$ where $f^{(5)}(z)$ is the maximum value that the 5^{th} derivative can take on the interval.

$$Error < \left|\frac{e^{\sin 1}}{(5)!}(1-0)^5\right| \to Error < 0.019$$

7. B (1997 BC17) DL: 5

$f(x) = \ln(3-x) \quad \to f(2) = 0$

$f'(x) = \frac{-1}{3-x} = \frac{1}{x-3} \quad \to f'(2) = -1$

$f''(x) = \frac{-1}{(x-3)^2} \quad \to f''(2) = -1$

$f'''(x) = \frac{2}{(x-3)^3} \quad \to f'''(2) = -2$

$$f(x) \approx -(x-2) - \frac{(x-2)^2}{2} - \frac{(x-2)^3}{3}$$

8. E (1998 BC14) DL: 3

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\sin 1 \approx 1 - \frac{1^3}{3!} + \frac{1^5}{5!} = 1 - \frac{1}{6} + \frac{1}{120}$$

9. D (2003 BC 28) DL: 4

$$f(x)=\frac{1}{(1+x)^2}=(1+x)^{-2}$$

$$f'(x)=-2(1+x)^{-3}$$

$$f''(x)=6(1+x)^{-4}=\frac{6}{(1+x)^4}\rightarrow f''(0)=6$$

The coefficient of x^2 is $\frac{f''(0)}{2!}=\frac{6}{2!}=3$

Alternative solution:

$$\frac{1}{1-x}=1+x+x^2+x^3+\ldots+x^n+\ldots \quad \text{and} \quad \frac{1}{1+x}=\frac{1}{1-(-x)}=1-x+x^2-x^3+\ldots+(-1)^n x^n+\ldots$$

$$\frac{1}{(1+x)^2}=\frac{1}{(1+x)}\cdot\frac{1}{(1+x)}=\left(1-x+x^2-x^3+\ldots\right)\left(1-x+x^2-x^3+\ldots\right)$$

$$=1-x+x^2-x^3+\ldots$$
$$-x+x^2-x^3+x^4\ldots$$
$$+x^2-x^3+x^4-x^5\ldots$$
$$-x^3+x^4-x^5+x^6\ldots$$

Adding x^2 terms yields $3x^2$

10. E (2008 BC23) DL: 4

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

$$\sin 2x=2x-\frac{8x^3}{3!}+\frac{32x^5}{5!}-\frac{128x^7}{7!}+\cdots$$

$$x\sin 2x=2x^2-\frac{8x^4}{3!}+\frac{32x^6}{5!}-\frac{128x^8}{7!}+\cdots$$

11. D (2003 BC20) DL: 5

A, B, and C can not be the answer because of the terms in the denominator.

Answer choice D can be written as $x^2e^x-x^3-x^2=x^2(e^x-x-1)$

$$\frac{x^4}{2!}+\frac{x^5}{3!}+\frac{x^6}{4!}+\ldots+\frac{x^{n+3}}{(n+1)!}+\ldots.=x^2\left(\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^{n+1}}{(n+1)!}+\ldots.\right)=x^2\left(e^x-x-1\right)$$

12. A (2003 BC 77) DL: 3

The coefficient of $x^3=\frac{f'''(0)}{3!}=-5\rightarrow f'''(0)=(-5)(3!)=-30$

13. A (2008 BC84) DL:4

$$T_3(x)=f(3)+\frac{f'(3)}{1!}(x-3)+\frac{f''(3)}{2!}(x-3)^2+\frac{f'''(3)}{3!}(x-3)^3=2+\frac{(-1)}{1!}(x-3)+\frac{(6)}{2!}(x-3)^2+\frac{(12)}{3!}(x-3)^3$$

$$=2-(x-3)+3(x-3)^2+2(x-3)^3$$

Free Response Solutions:
14. 2008 BC3

(a) $P_1(x) = 80 + 128(x-2)$, so $h(1.9) \approx P_1(1.9) = 67.2$ $P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.	4 { 2: $P_1(x)$ 1: $P_1(1.9)$ 1: $P_1(1.9) < h(1.9)$ with reason
(b) $P_3(x) = 80 + 128(x-2) + \frac{488}{6}(x-2)^2 + \frac{488}{18}(x-2)^3$ $h(1.9) \approx P_3(1.9) = 67.988$	3 { 2: $P_3(x)$ 1: $P_3(1.9)$
(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\max\limits_{1.9 \le x \le 2} \left\| h^{(4)}(x) \right\| = \frac{584}{9}$. Therefore, $\left\| h(1.9) - P_3(1.9) \right\| \le \frac{584}{9} \frac{\left\|1.9-2\right\|^4}{4!}$ $= 2.7037 \times 10^{-4}$ $< 3 \times 10^{-4}$	2 { 1: form of Lagrange error estimate 1: reasoning

15. 2007 BC6

(a) $e^{-x^2} = 1 + \frac{(-x^2)}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots$

$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$

3 { 1: two of $1,\ -x^2,\ \frac{x^4}{2},\ -\frac{x^6}{6}$
1: remaining terms
1: general terms }

(b) $\frac{1 - x^2 - f(x)}{x^4} = -\frac{1}{2} + \frac{x^2}{6} + \sum_{n=4}^{\infty} \frac{(-1)^{n+1} x^{2n-4}}{n!}$

Thus, $\lim_{x \to 0} \left(\frac{1 - x^2 - f(x)}{x^4} \right) = -\frac{1}{2}$.

1: answer

(c) $\int_0^x e^{-t^2}\, dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots \right) dt$

$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$

Using the first two terms of this series, we estimate that $\int_0^{1/2} e^{-t^2}\, dt \approx \frac{1}{2} - \left(\frac{1}{3}\right)\left(\frac{1}{8}\right) = \frac{11}{24}$.

3 { 1: two terms
1: remaining terms
1: estimate }

(d) $\left| \int_0^{1/2} e^{-t^2}\, dt - \frac{11}{24} \right| < \left(\frac{1}{2}\right)^5 \cdot \frac{1}{10} = \frac{1}{320} < \frac{1}{200}$, since

$\int_0^{1/2} e^{-t^2}\, dt = \sum_{n=4}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{n!(2n+1)}$, which is an alternating series with individual terms that decrease in absolute value to 0.

2 { 1: uses the third term as the error bound
1: explanation }

16. 2005 BC6

(a) $P_6(x) = 7 + \frac{1!}{3^2} \cdot \frac{1}{2!}(x-2)^2 + \frac{3!}{3^4} \cdot \frac{1}{4!}(x-2)^4 + \frac{5!}{3^6} \cdot \frac{1}{6!}(x-2)^6$

3 {
1: polynomial about $x = 2$
2: $P_6(x)$
$\langle -1 \rangle$ each incorrect term
$\langle -1 \rangle$ max for all extra terms, $+\cdots$, misuse of equality
}

(b) $\frac{(2n-1)!}{3^{2n}} \cdot \frac{1}{(2n)!} = \frac{1}{3^{2n}(2n)}$

1: coefficient

(c) The Taylor series for f about $x = 2$ is

$$f(x) = 7 + \sum_{n=1}^{\infty} \frac{1}{2n \cdot 3^{2n}}(x-2)^{2n}.$$

$$L = \lim_{n\to\infty} \left| \frac{\frac{1}{2(n+1)} \cdot \frac{1}{3^{2(n+1)}}(x-2)^{2(n+1)}}{\frac{1}{2n} \cdot \frac{1}{3^{2n}}(x-2)^{2n}} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{2n}{2(n+1)} \cdot \frac{3^{2n}}{3^2 3^{2n}}(x-2)^2 \right| = \frac{(x-2)^2}{9}$$

$L < 1$ when $|x-2| < 3$.

Thus, the series converges when $-1 < x < 5$.

When $x = 5$, the series is $7 + \sum_{n=1}^{\infty} \frac{3^{2n}}{2n \cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, diverges.

When $x = -1$, the series is $7 + \sum_{n=1}^{\infty} \frac{(-3)^{2n}}{2n \cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, diverges.

The interval of convergence is $(-1, 5)$.

5 {
1: set up ratio
1: computes limit of ratio
1: identifies interior of interval of convergence
1: considers both endpoints
1: analysis/conclusion for both endpoints
}

17. 2015 BC Question 6

(a) Let a_n be the nth term of the Maclaurin series. $\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \bullet \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} \bullet x$ $\lim_{n\to\infty} \left\| \frac{-3n}{n+1} \bullet x \right\| = 3\|x\|$ $3\|x\| < 1 \Rightarrow \|x\| < \frac{1}{3}$ The radius of convergence is $R = \frac{1}{3}$.	3: { 1: sets up ratio; 1: computes limit of ratio; 1: determines radius of convergence }
(b) The first four nonzero terms of the Maclaurin series for $f\,'$ are $1 - 3x + 9x^2 - 27x^3$. $f\,'(x) = \frac{1}{1-(-3x)} = \frac{1}{1+3x}$	3: { 2: first four nonzero terms; 1: rational function }
(c) The first four nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. The product of the Maclaurin series for e^x and the Maclaurin series for f is $\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\left(x - \frac{3}{2}x^2 + 3x^3 - \ldots\right)$ $= x - \frac{1}{2}x^2 + 2x^3 + \ldots$ The third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$ is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$.	3: { 1: first four nonzero terms of the Maclaurin series for e^x; 2: Taylor polynomial }