NATIONAL MATH + SCIENCE INITIATIVE

AP Calculus

Taylor Series

Student Handout

2017-2018 EDITION

Taylor Series

Students should be able to:

- Construct and use Taylor polynomials.
- Write a power series representing a given function.
- Derive a power series for a given function by various methods (e.g. algebraic processes, substitutions, using properties of geometric series, and operations on known series such as term by term integration or term by term differentiation).
- Determine the radius and interval of convergence of a power series.
- Use Lagrange error bound and in some cases where the signs of a Taylor polynomial are alternating, use the alternating series error bound to bound the error of a Taylor polynomial approximation to a function.

Multiple Choice:

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	-2	1	4
1	2	-3	3	-2
2	-1	1	4	5

1. (calculator not allowed)

Selected values of a function f and its first three derivatives are indicated in the table above. What is the third degree Taylor polynomial for f about $x = 1$?

(A) $2 - 3x + \frac{3}{2}x^2 - \frac{1}{3}x^3$

(B) $2 - 3(x-1) + \frac{3}{2}(x-1)^2 - \frac{1}{3}(x-1)^3$

(C) $2 - 3(x-1) + \frac{3}{2}(x-1)^2 - \frac{2}{3}(x-1)^3$

(D) $2 - 3(x-1) + 3(x-1)^2 - 2(x-1)^3$

2. (calculator not allowed)
The third-degree Taylor polynomial for the function f about $x = 0$ is

$$T(x) = 3 - 4x + 2x^2 - 3x^3$$

Which of the following tables gives the values of f and its first three derivatives at $x = 0$?

(A)

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	-8	6	-12

(B)

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	-4	2	-3

(C)

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	-4	4	-18

(D)

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
0	3	-4	4	-9

3. (calculator not allowed)

What is the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(x-4)^{2n}}{3^n}$?

(A) $2\sqrt{3}$
(B) 3
(C) $\sqrt{3}$
(D) $\frac{\sqrt{3}}{2}$
(E) 0

4. (calculator not allowed)

For $x > 0$, the power series $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \cdots$

(A) $\cos x$
(B) $\sin x$
(C) $\frac{\sin x}{x}$
(D) $e^x - e^{x^2}$
(E) $1 + e^x - e^{x^2}$

5. (calculator not allowed)

What is the coefficient of x^4 in the Taylor series for $\cos^2 x$ about $x = 0$?

(A) $-\frac{1}{2}$
(B) $\frac{1}{8}$
(C) $\frac{1}{3}$
(D) $\frac{3}{4}$

6. (calculator allowed)
   The function f has derivatives of all orders for all real numbers, and $f^{(5)}(x) = e^{\sin x}$. If the fourth-degree Taylor polynomial for f about $x = 0$ is used to approximate f on the interval $[0,1]$, what is the Lagrange error bound for the maximum error on the interval $[0,1]$?

   (A) 0.008
   (B) 0.019
   (C) 0.023
   (D) 0.025

7. (calculator not allowed)
   Let f be the function given by $f(x) = \ln(3-x)$. The third-degree Taylor polynomial for f about $x = 2$ is

   (A) $-(x-2)+\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}$
   (B) $-(x-2)-\frac{(x-2)^2}{2}-\frac{(x-2)^3}{3}$
   (C) $(x-2)+(x-2)^2+(x-2)^3$
   (D) $(x-2)+\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}$
   (E) $(x-2)-\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}$

8. (calculator not allowed)
   What is the polynomial approximation for the value of sin 1 obtained by using the fifth-degree Taylor polynomial about $x = 0$ for $\sin x$?

   (A) $1-\frac{1}{2}+\frac{1}{24}$
   (B) $1-\frac{1}{2}+\frac{1}{4}$
   (C) $1-\frac{1}{3}+\frac{1}{5}$
   (D) $1-\frac{1}{4}+\frac{1}{8}$
   (E) $1-\frac{1}{6}+\frac{1}{120}$

9. (calculator not allowed)

What is the coefficient of x^2 in the Taylor series for $\dfrac{1}{(1+x)^2}$ about $x=0$?

(A) $\dfrac{1}{6}$
(B) $\dfrac{1}{3}$
(C) 1
(D) 3
(E) 6

10. (calculator not allowed)
If $f(x)=x\sin(2x)$, which of the following is the Taylor series for f about $x=0$?

(A) $x-\dfrac{x^3}{2!}+\dfrac{x^5}{4!}-\dfrac{x^7}{6!}+\ldots$
(B) $x-\dfrac{4x^3}{2!}+\dfrac{16x^5}{4!}-\dfrac{64x^7}{6!}+\ldots$
(C) $2x-\dfrac{8x^3}{3!}+\dfrac{32x^5}{5!}-\dfrac{128x^7}{7!}+\ldots$
(D) $2x^2-\dfrac{2x^4}{3!}+\dfrac{2x^6}{5!}-\dfrac{2x^8}{7!}+\ldots$
(E) $2x^2-\dfrac{8x^4}{3!}+\dfrac{32x^6}{5!}-\dfrac{128x^8}{7!}+\ldots$

11. (calculator not allowed)

A function f has Maclaurin series given by $\dfrac{x^4}{2!}+\dfrac{x^5}{3!}+\dfrac{x^6}{4!}+\ldots+\dfrac{x^{n+3}}{(n+1)!}+\ldots$. Which of the following is an expression for $f(x)$?

(A) $-3x\sin x+3x^2$
(B) $-\cos(x^2)+1$
(C) $-x^2\cos x+x^2$
(D) $x^2e^x-x^3-x^2$
(E) $e^{x^2}-x^2-1$

12. (calculator allowed)

Let $P(x) = 3x^2 - 5x^3 + 7x^4 + 3x^5$ be the fifth-degree Taylor polynomial for the function f about x. What is the value of $f'''(0)$?

(A) –30
(B) –15
(C) –5
(D) $-\frac{5}{6}$
(E) $-\frac{1}{6}$

13. (calculator allowed)
Let f be a function with $f(3) = 2,$ $f'(3) = -1,$ $f''(3) = 6,$ and $f'''(3) = 12.$ Which of the following is the third-degree Taylor polynomial for f about $x = 3$?

(A) $2 - (x-3) + 3(x-3)^2 + 2(x-3)^3$
(B) $2 - (x-3) + 3(x-3)^2 + 4(x-3)^3$
(C) $2 - (x-3) + 6(x-3)^2 + 12(x-3)^3$
(D) $2 - x + 3x^2 + 2x^3$
(E) $2 - x + 6x^2 + 12x^3$

Free Response :

14. (calculator allowed)

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values for h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater or less than $h(1.9)$? Explain your answer.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with an error less than 3×10^{-4}.

15. (calculator not allowed)

Let f be the function given by $f(x) = e^{-x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use your answer from part (a) to find $\lim_{x \to 0} \frac{1 - x^2 - f(x)}{x^4}$.

(c) Write the first four nonzero terms of the Taylor Series for $\int_0^x e^{-t^2}\, dt$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}\, dt$.

(d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\, dt$ by less than $\frac{1}{200}$.

16. (calculator not allowed)
Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative at $x = 2$ is given by $f^{(n)}(2) = \dfrac{(n-1)!}{3^n}$.

(a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

(b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

(c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

17. (calculator not allowed)
The Maclaurin series for a function *f* if given by
$\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{n}x^n = x - \frac{3}{2}x^2 + 3x^3 - \ldots + \frac{(-3)^{n-1}}{n}x^n + \ldots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of *f*. Express f' as a rational function for $|x| < R$.

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

Taylor Series Reference

Taylor series provide a way to find a polynomial "look-alike" to a non-polynomial function. This is done by a specific formula shown below (which should be memorized).

Taylor Series centered at $x = a$

Let f be a function with derivatives of all orders on an interval containing $x = a$. Then f, centered at $x = a$, can be represented by

$$f(x) = \frac{f(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

When $a = 0$, f is centered at the origin which is a special case called the MacLaurin series. Generally, it is not necessary to simplify results on the Free Response section. Answers will be simplified on the Multiple Choice section.

The formulas (recommended to be memorized) are as follows:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty \text{ is the interval of convergence.}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}; \quad -\infty < x < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}; \quad -\infty < x < \infty$$

Geometric power series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n; \; -1 < x < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n; \; -1 < x < 1$$

There are three main types of questions asked on the exam:

- Write a function in terms of a series or match a series to an appropriate function
- Find an error bound on an *n*th degree Taylor Polynomial
- Find an interval or radius of convergence

Error Bounds
To determine an error bound for a Taylor polynomial, first classify the polynomial as either an alternating or non-alternating series. Their error bounds are found as follows:

Alternating Series
When a series is alternating, the error is maximized in the next unused term evaluated at the difference between the center of the convergence and the x-coordinate being evaluated.

Non-Alternating Series
If a series is non-alternating, the error is still tied up in the next term by the formula $Error < \left| \frac{f^{(n+1)}(z)}{(n+1)!}\left(x-a\right)^{n+1} \right|$ where $f^{(n+1)}(z)$ is the maximum value that the *(n+1)* derivative can take on the interval.

Interval of Convergence for Taylor Series
When looking for the interval of convergence for a Taylor Series, refer back to the interval of convergence for each of the basic Taylor Series formulas. Fit your function to the function being tested.

Sometimes, the exam will manipulate a Taylor series to a power series before asking for the interval of convergence. The most common test to find the interval of convergence for a power series is the Ratio Test, which says that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$. If $L < 1$, the series converges. If $L > 1$, the series diverges. If $L = 1$, the test fails and another test should be used. When using the Ratio Test, it is important to remember that the Ratio Test only checks the *open* interval. The endpoints of the interval must be checked separately to determine if the interval is open or closed. If a series is known to be geometric, the endpoints do not need to be checked since convergence requires $|r| < 1$; therefore the endpoints cannot be included.