NATIONAL MATH + SCIENCE INITIATIVE

AP Calculus

Fundamental Theorem of Calculus

Presenter Notes

2016-2017 EDITION

Student Study Session - Presenter Notes

Thank you for agreeing to present at one of NMSI's Saturday Study Sessions. We are grateful you are sharing your time and expertise with our students. Saturday mornings can be a "tough sell" for students, so we encourage you to incorporate strategies and techniques to encourage student movement and engagement. Suggestions for different presentation options are included in this document. If you have any questions about the content or about presentation strategies, please contact Mathematics Director Charla Holzbog at cholzbog@nms.org or AP Calculus Content Specialist Karen Miksch at kmiksch@nms.org.

The material provided contains many released AP multiple choice and free response questions as well as some AP-like questions that we have created. The goal for the session is to let the students experience a variety of both types of questions to gain insight on how the topic will be presented on the AP exam. It is also beneficial for the students to hear a voice other than their teacher in order to help clarify their understanding of the concepts.

Suggestions for presenting:
The vast majority of the study sessions are on Saturday and students and teachers are coming to be WOWed! We want activities to engage the students as well as prepare them for the AP Exam. The following presenter notes include pacing suggestions (you only have 50 minutes!), solutions, and recommended engagement strategies.

Suggestions on how to prepare:

- The notes/summaries on the last page(s) are for reference. We want the students' time during the session to be focused on the questions as much as possible and not taking or reading the notes. As the questions are presented during the session, you may wish to refer the students back to those pages as needed. It is not our intent for the sessions to begin with a lecture over these pages.
- As you prepare, work through the questions in the packet noting the level of difficulty and topic or skill required for the questions.
- Design a plan for what questions you would like to cover with the group depending on their level of expertise. Some groups will be ready for the tougher questions while other groups will need more guidance and practice on the easier ones. Create an easy, medium, and hard listing of the questions prior to the session. This will allow you to adjust on the fly as you get to know the groups. In most instances, there will **not** be enough time to cover all the questions in the packet. Use your judgement on the amount of questions to cover based on the students' interactions. Remember to include both multiple choice and free response type questions. Discussions on test taking strategies and scoring of the free response questions are always great to include during the day.
- The concepts should have been previously taught; however, be prepared to "teach" the topic if you find out the students have not covered the concept prior in class. In sessions where multiple schools come together, you might have a mixture of students with and without prior knowledge on the topic. You will have to use your best judgement in this situation.
- Consider working through some free response questions before the multiple choice questions, or flipping back and forth between the two types of questions. Sometimes, if free response questions are saved for the last part of the session, it is possible students only get practice with one or two of them and most students need additional practice with free response questions.

Fundamental Theorem of Calculus Student Session-Presenter Notes

This session includes a reference sheet at the back of the packet. We suggest that the presenter not spend time going over the reference sheet, but point it out to students so that they may refer to it if needed.

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use our presenter notes and your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted. Notice in the solutions guide the questions are categorized as 3, 4, or 5 indicating a typical question of the difficulty level (DL) for a student earning these qualifying scores on the AP exam.

The students will participate in a "speed dating" activity during this session, so arrange the desks in pairs before the students arrive.

I. 5 Minutes: Introductory Activity

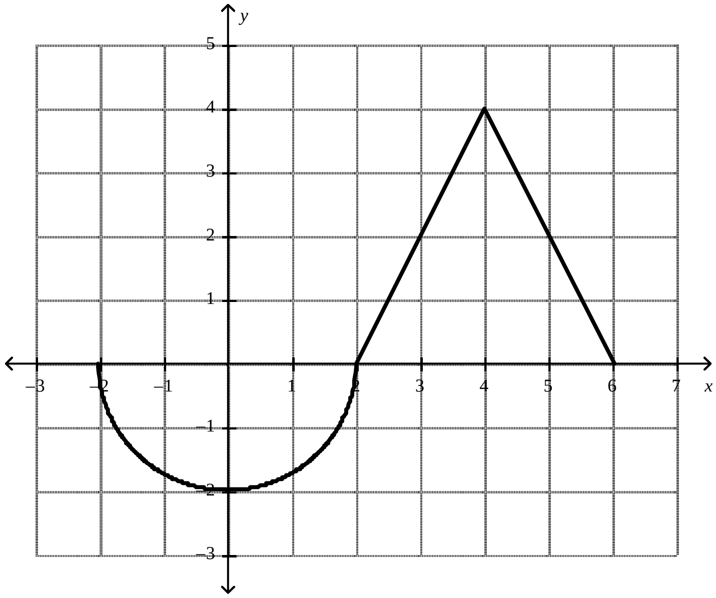
- Before beginning the questions in the packet, display the graph and questions on the following page for the students. Instruct the students to work together on these questions while you walk around to monitor their progress and listen to their conversations. Use this time to gauge the level of knowledge of the group.
- After a few minutes, discuss the answers or allow students to come to the board to explain their process. Try not to spend more than 5 minutes on this activity.

II. 25 minutes: Multiple Choice Practice --*your choice of questions depending on the comfort level of the students during the introductory activity —some suggested questions are listed below.*

- The students will practice multiple choice questions in pairs in a speed dating activity. The students should be sitting in pairs. Assign a question for the students to work and allow 1-2 minutes for the students to work the question with their "date" before reviewing the answer. After each question, the student on the right will remain in the same seat, while the student on the left rotates so that they have a new "date" for every question. Continue this process, working through as many multiple choice questions as you can in 25 minutes. A suggested order of questions is: 1, 3, 4, 6, 7, 8, 9, 11, 12, 15, 16. Include questions 2, 5, and 13 with stronger groups.

III. 20 minutes: Free Response Practice

- Continue the speed dating activity to practice the free response questions. Allow an appropriate amount of time for the students to work each questions with their "date" before reviewing the answers. A suggested order of questions is: 18, 20, 23, 21, 24.



Graph of f'

The graph of f' consists of a semicircle and two line segments as shown above.

Given that $f(0) = 4$, find:

(a) $f(2)$

(b) $f(6)$

(c) $f(-2)$

Introductory Activity

(a) $f(2)=f(0)+\int_0^2 f'(x)dx=4-\pi$

(b) $f(6)=f(0)+\int_0^6 f'(x)dx=4-\pi+8=12-\pi$

(c) $f(-2)=f(0)-\int_{-2}^0 f'(x)dx=4-(-\pi)=4+\pi$

Multiple Choice Questions Solutions

1. B (AP-like) DL: 3

   $t^4-t^2\Big|_{-2}^{x}=x^4-x^2-\left[(-2)^4-(-2)^2\right]=x^4-x^2-12$

2. C (1988 AB36) DL: 4

   The first quadrant interval for $y=3x-x^2$ is [0, 3].

   $$y_{avg}=\frac{\int_0^3\left(3x-x^2\right)dx}{3-0}$$

   $$y_{avg}=\frac{\left(\frac{3}{2}x^2-\frac{1}{3}x^3\right)\Big|_{x=0}^{x=3}}{3}$$

   $$y_{avg}=\frac{\left(\frac{27}{2}-9\right)-0}{3}=\frac{3}{2}$$

3. D (2003 AB22) DL: 3

   $f(1)=f(0)+\int_0^1 f'(x)\,dx=5+3=8$

   Alternatively, the equation for the derivative shown is $f'(x)=-6x+6$.

   $f(x)=\int(-6x+6)dx$

   $f(x)=-3x^2+6x+c$

   With $f(0)=5$ implies $c=5$ and therefore $f(1)=3(1)^2-6(1)+5=8$

4. E (2003 AB23) DL: 3

   Applying the Second Fundamental Theorem, $\frac{d}{dx}\int_a^{g(x)} f(t)dt=f(g(x))\,g'(x)$

   $\frac{d}{dx}\int_0^{x^2}\sin(t^3)dt=\left(\sin(x^2)^3\right)(2x)=2x\sin(x^6)$

5. C (1993 BC41 appropriate for AB) DL: 5

$$f'(x)=\frac{d}{dx}\int_{-2}^{x^2-3x} e^{t^2}\,dt = e^{(x^2-3x)^2}(2x-3)$$

$f'(x)=0$ only when $x=\dfrac{3}{2}$.

$f'<0$ for $\left(-\infty,\dfrac{3}{2}\right)$ and $f'>0$ for $\left(\dfrac{3}{2},\infty\right)$ thus $f(x)$ has a minimum at $x=\dfrac{3}{2}$.

6. D (1988 AB14) DL: 4

$$\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{1+\sin\theta}}\,d\theta$$

Let $u=1+\sin\theta$

$$\int_1^2 u^{\frac{-1}{2}}\,du = 2u^{\frac{1}{2}}\Big|_{u=1}^{u=2} = 2\sqrt{2}-2$$

7. C (2003 BC18 appropriate for AB) DL: 3

$$g'(x)=\frac{d}{dx}\int_0^{2x} f(t)\,dt = 2f(2x)$$

$$g'(3)=2f(2\cdot 3)=2f(6)=2(-1)=-2$$

8. D (AB Sample Question #20 from AP Calculus Course and Exam Description) DL: 3

$$g(4)=g(2)+\int_2^4 h(x)dx$$

$$g(4)=3+0.15153=3.152$$

9. C (AP-like) DL: 4

$$y(1)=2-\int_1^1 e^{t^2}\,dt=2$$

$$\frac{dy}{dx}=-e^{x^2}\Big|_{x=1}=-e$$

Therefore, $y=-e(x-1)+2$

10. D (AB Sample Question #7 from 2014 Fall AP Calculus Curriculum Framework) DL: 3

$$f'(x)=\frac{3x^2}{1+\ln\left(x^3\right)}$$

$$f'(2)=\frac{3(2)^2}{1+\ln\left(2^3\right)}=\frac{12}{1+\ln 8}$$

11. A (AP-like) DL: 3

$$f(3)=f(5)-\int_3^5 \ln\left(x^2\right)dx$$

$$f(3)=8-5.50705=2.497$$

12. D (2003 AB92) DL: 3

$g(x)$ is decreasing when $g'(x) < 0$.

$g'(x) = \frac{d}{dx}\int_0^x \sin(t^2)dt$

$g'(x) = \sin(x^2)$

Using a graphing calculator, determine where $g'(x) < 0$.

$1.772 \le x \le 2.507$

13. B (1997 BC82 appropriate for AB) DL: 4

Since $\int_0^x (t^2 - 2t)\, dt \ge \int_2^x t\, dt$, $\frac{1}{3}x^3 - x^2 \ge \frac{1}{2}x^2 - 2$. Using the calculator, the greatest x-value on the interval $[0, 4]$ that satisfies this inequality is found to occur at $x = 1.3887$.

14. E (1997 AB88) DL: 3

$f(x) = \int_a^x h(x)dx$, so $f(a) = 0$; therefore, only choices (A) and (E) are possible. But $f'(x) = h(x)$, so f is differentiable at $x = b$. This is true for the graph in option (E), but not for the graph in option (A), where there appears to be a sharp turn at $x = b$. Also, since h is increasing at first, the graph of f must start out concave up. This is also true in (E) but not (A).

15. A (2003 AB84) DL: 4

$T(5) = T(0) + \int_0^5 (-110e^{-0.4t})\, dt = 350 - 237.78 \approx 112^\circ F$

16. A (2008 BC88 appropriate for AB) DL: 4

$g(x) = \int_2^x f(t)dt$ and $f > 0$ and $f' < 0$; $g'(x) = f(x)$, so $g''(x) = f'(x)$. Since $g'(x) = f(x)$ is given to be positive and decreasing, g is increasing and concave down, so C,D, and E can be eliminated since they represent decreasing functions. $g(2) = \int_2^2 f(t)\, dt = 0$. Since $f > 0,\ g' > 0$, so g is increasing, so $f(1) < g(2)$, so $g(1) = -2$. $f' < 0$, so $g'' < 0$, so g' is decreasing; therefore, $\frac{g(3) - g(2)}{3 - 2} < \frac{g(2) - g(1)}{2 - 1}$, so the answer is choice A.

17. A (2003 AB82) DL: 4

The function $r(t)$ is the rate of change in the altitude, so the altitude is decreasing when $r(t) < 0$. The zeros of $r(t)$ are 1.572 and 3.514, so the change in altitude when the altitude is decreasing can be found using $\int_{1.572}^{3.514} r(t)dt$.

Free Response Questions
18. 2009 AB5/BC5b

(b) $\int_2^{13}\left(3-5f'(x)\right)dx=\int_2^{13}3\,dx-5\int_2^{13}f'(x)\,dx$ $=3(13-2)-5\left(f(13)-f(2)\right)=8$	$2\begin{cases}1: & \text{uses Fundamental Theorem of Calculus}\\ 1: & \text{answer}\end{cases}$

19. 2010 AB5a

(a) $g(3)=5+\int_0^3 g'(x)\,dx=5+\frac{\pi\cdot 2^2}{4}+\frac{3}{2}=\frac{13}{2}+\pi$ $g(-2)=5+\int_0^{-2} g'(x)\,dx=5-\pi$	$3\begin{cases}1: & \text{uses } g(0)=5\\ 1: & g(3)\\ 1: & g(-2)\end{cases}$

20. 2011 AB/BC2cd

(c) $\int_0^{10}H'(t)\,dt=H(10)-H(0)=43-66=-23$ The temperature of the tea drops 23 degrees Celsius from time $t=0$ to time $t=10$ minutes.	$2\begin{cases}1: & \text{value of integral}\\ 1: & \text{meaning of expression}\end{cases}$
(d) $B(10)=100+\int_0^{10}B'(t)\,dt=34.18275;\; H(10)-B(10)=8.817$ The biscuits are 8.817 degrees Celsius cooler than the tea.	$3\begin{cases}1: & \text{integrand}\\ 1: & \text{uses } B(0)=100\\ 1: & \text{answer}\end{cases}$

21. 2009B AB1ac

(a) $R(t)=6+\int_0^t \frac{1}{16}\left(3+\sin\left(x^2\right)\right)dx$ $R(3)=6.610$ or 6.611	$3\begin{cases}1: & \text{integral}\\ 1: & \text{expression for } R(t)\\ 1: & R(3)\end{cases}$
(c) $\int_0^3 A'(t)\,dt=A(3)-A(0)=24.200$ or 24.201 From time $t=0$ to $t=3$ years, the cross-sectional area grows by 24.201 square centimeters.	$3\begin{cases}1: & \text{uses Fundamental Theorem of Calculus}\\ 1: & \text{value of } \int_0^3 A'(t)\,dt\\ 1: & \text{meaning of } \int_0^3 A'(t)\,dt\end{cases}$

22. 2011B AB6 ac

(a) $\int_{-2\pi}^{4\pi} f(x)\,dx = \int_{-2\pi}^{4\pi}\left(g(x) - \cos\left(\frac{x}{2}\right)\right)dx$ $= 6\pi^2 - \left[2\sin\left(\frac{x}{2}\right)\right]_{x=-2\pi}^{x=4\pi}$ $= 6\pi^2$	2 { 1: antiderivative 1: answer
(c) $h'(x) = g(3x)\cdot 3$ $h'\left(-\frac{\pi}{3}\right) = 3g(-\pi) = 3\pi$	3 { 2: $h'(x)$ 1: answer

23. 2014 AB/BC3b

(b) $g'(x) = f(x)$ The graph of g is increasing and concave down on the intervals $-5 < x < -3$ and $0 < x < 2$ because $g' = f$ is positive and decreasing on these intervals.	2 { 1: answer 1: reason

24. 2015 AB/BC1 ac

(a) $\int_0^8 R(t)dt = 75.570$

2: { 1: integrand
1: answer }

(c) The amount of water in the pipe at time t,

$0 \le t \le 8$, is $30 + \int_0^t \left[R(x) - D(x)\right] dx.$

R(t) – D(t) = 0 $\Rightarrow$ t = 0, 3.271658

t	Amount of water in the pipe
0	30
3.271658	27.964561
8	48.543686

The amount of water in the pipe is minimum at time t= 3.272 (or 3.271) hours

3: { 1: considers R(t) –D (t) =0
1: answer
1: justification }

Fundamental Theorem of Calculus Reference Page

- Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

- The Fundamental Theorem of Calculus can be written in various ways.

$$\int_a^b f'(x)dx = f(b) - f(a)$$

$$f(b) = f(a) + \int_a^b f'(x)dx$$

$$f(a) = f(b) - \int_a^b f'(x)dx$$

$$\int_a^b f''(x)dx = f'(b) - f'(a)$$

- Average value of a function over a particular interval

$$f_{avg} = \frac{\int_a^b f(x)dx}{b-a} = \frac{F(b) - F(a)}{b-a}$$

- Other Fundamental Theorem of Calculus

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

$$\frac{d}{dx}\left(\int_a^{g(x)} f(t)dt\right) = f(g(x))\,g'(x)$$

- Fundamental Theorem of Calculus in context

$$Amount = \int_{begining\ time}^{ending\ time} Rate\ dt$$

$$Current\ Amount = Initial\ Amount + \int_{time1}^{time2} "rate\ in"\ dt - \int_{time1}^{time2} "rate\ out"\ dt$$